1994

Bounded Linear Operators On Banach Sequence Spaces

Xiaopeng Gao

BOUNDED LINEAR OPERATORS ON BANACH SEQUENCE SPACES

by

XIAOPENG GAO

Department of Mathematics

Submitted in partial fulfilment

of the requirements for the degree of

Doctor of Philosophy

Faculty of Graduate Studies

The University of Western Ontario

London, Ontario

April, 1994

ISBN 0-315-93176-0

Abstract

We investigate matrices and sequences of operators as bounded linear operators on Banach sequence spaces in various situations, and some topics related to these matrices and sequences. This thesis consists of five chapters.

In the first chapter we study whether an infinite matrix, particularly a summability matrix, is a bounded linear operator on l_p ($p \geq 1$). Some restrictive conditions for Nörlund and weighted mean matrices to be in $B(l_p)$ imposed by earlier authors are eliminated. Some results for weighted mean matrices are proved as consequences of more general results for generalized Hausdorff matrices.

A necessary and sufficient condition for a non-negative matrix to map l_p to l_q ($1 \leq q \leq p < \infty$) is refined in the second chapter, and a generalized Vere-Jones conjecture related to this problem is solved.

The third chapter is on the topic of subalgebras Γ_w and Ω_w of $B(X)$, where X is a nonreflexive Banach space and $w \in X^{**}$. These algebras arise as generalizations of the classical algebra of 'conservative matrices', i.e., matrices that are in $B(c)$. Algebraic and set-theoretic relationships between these subalgebras are studied. The relationships among such subalgebras of $B(c_0)$ are clarified. For $B(l_1)$, the subalgebras associated with any Dirac measure, i.e., a unit point mass, are found to be isomorphic, but never equal, to those associated with some Banach limit, i.e., a translation invariant extended limit. With the aid of the Stone-Čech compactification of **N**,

the intersections of all Γ's or all Ω's associated with Banach limits or Dirac measures are characterized.

In chapter 4, we investigate operators on sequence spaces with terms in a Banach space. Some classes of transformations, e.g., those between spaces of convergent sequences are characterized. In the course of the work various questions about weak, norm, unconditional and absolute convergence arise and are discussed.

The last chapter is concerned with generalized Kōthe-Toeplitz duals of Banach sequence spaces. The relationship between the various duals of $c_0(X)$, $c(X)$, $l_\infty(X)$, and $bv(X)$ are examined.

献给我的母亲和父亲

Acknowledgement

The thesis is written under the supervision of Dr. F. P. Cass. The study was done with help and instruction from Dr. Cass and Dr. D. Borwein, who was my supervisor when Dr. Cass was on sabbatical leave. I am grateful for and will forever remember the patience and insights they showed in the guidance of my work.

I am grateful to the Natural Sciences and Engineering Research Council of Canada for its generous support of my work through a Postgraduate Scholarship.

I deeply appreciate the help from Dr. G. Sinnamon, Dr. P. Milnes, and Dr. A. Boivin in my research and graduate work, from Dr. L. Renner and Dr. A. Heinicke in arranging my study program, and from Dr. S. Rankin and Dr. M. Dawes for overcoming my difficulties in using TEX. I would like to extend my thanks to other faculty members in the Department of Mathematics for broadening my mathematical knowledge. Acknowledgement is also due to Ms. J. Williams, Ms. K. Martin, Ms. T. Slivinsky, and Mrs. D. Mayea, for their secretarial assistance.

Last but not least, my thanks go to my dear wife, Xin Wang. This thesis would never be possibly written without her constant support throughout the years.

Contents

Chapter 1

Summability Matrices as Operators on l_p

1.1 Introduction

All sequences $x = \{x_k\}$ that we use will be sequences with $x_k \in \mathbb{C}$. In this chapter the indices n and k range from 0 to ∞, whereas in other chapters they range from 1 to ∞.

By a sequence space X we mean a linear space over $\mathbb{C}$ whose elements are sequences with terms in $\mathbb{C}$. The operations of addition and scalar multiplication are defined as follows. For $x = \{x_k\}$, $y = \{y_k\} \in X$ and $\lambda \in \mathbb{C}$,

$$x + y = \{x_k + y_k\}, \qquad \text{and}$$

$$\lambda x = \{\lambda x_k\}.$$

We shall write e for the sequence $\{1, 1, 1, \ldots\}$, and e_n $(n \geqslant 0)$ for the sequence $\{\delta_{nk}\}_{k=0}^{\infty}$ where

$$\delta_{nk} = \begin{cases} 1 & \text{if } n = k \\ 0 & \text{if } n \neq k. \end{cases}$$

An FK-space is a sequence space which is also a Fréchet space and for which the maps $P_n : \{x_k\} \mapsto x_n$ are continuous for $n = 0, 1, 2, \ldots$. Some properties of FK-spaces are developed in [48, §11.3]. An FK-space which is a Banach space is referred to as a BK-space. We make frequent use of the following sequence spaces, each of which is a BK-space.

$$c_0 = \{\{x_k\} \mid \lim_k x_k = 0\},$$

$$c = \{\{x_k\} \mid \lim x = \lim_k x_k \text{ exists }\},$$

$$l_\infty = \{\{x_k\} \mid \sup_k |x_k| < \infty\}.$$

The norm of $x = \{x_k\}$ is $\|x\|_\infty = \sup_k |x_k|$ in the above three spaces, and for the space

$$l_p = \{\{x_k\} \mid \sum_k |x_k|^p < \infty\}, \qquad \text{for } 1 \leqslant p < \infty,$$

the norm of $x = \{x_k\}$ is $\|x\|_p = (\sum_k |x_k|^p)^{1/p}$.

By a matrix we shall understand an infinite matrix $A = (a_{nk})$ with $a_{nk} \in \mathbb{C}$. Given a matrix $A = (a_{nk})$ and a sequence $x = \{x_k\}$, we write formally,

$$y_n = A_n(x) = \sum_{k=0}^{\infty} a_{nk} x_k. \tag{1.1}$$

Suppose X and Y are sequence spaces. If the series $\sum_{k=0}^{\infty} a_{nk} x_k$ converges for every $x = \{x_k\} \in X$ and for each $n = 0, 1, 2, \ldots$, and if the sequence $A(x) = \{A_n(x)\} \in Y$ for each $x \in X$, we say $A \in (X, Y)$. Note that if a matrix $A \in (X, Y)$, then (1.1) defines a linear mapping from X to Y. If both X and Y are FK-spaces, it is a consequence of the Banach-Steinhaus theorem (see [48, p.117]) that the linear operator from X to Y defined by (1.1) is continuous.

On the other hand, the spaces c_0 and l_p $(1 \leqslant p < \infty)$ have $\{e_n\}_{n=0}^{\infty}$ as a Schauder basis (as defined in [26, II.5, pp.93]). Thus if X and Y are any of the spaces c_0 or l_p, $1 \leqslant p < \infty$, and $T \in B(X, Y)$, it is easy to see that T is given by the matrix $A = (a_{nk})$ in accordance with (1.1) where $a_{nk} = P_n T e_k$.

For $1 \leqslant p, q < \infty$ and $T \in B(l_p, l_q)$, the norm of T will be denoted by $\|T\|_{p,q}$ and by $\|T\|_p$ if $q = p$.

Definition 1.1 *If $A \in (c, c)$, A is called a conservative matrix. If in addition, $\lim Ax = \lim x$ for all $x \in c$, A is called a* **regular** *matrix.*

The following is the classical Silverman-Toeplitz Theorem (see [29, §3.2, Theorem 1]), giving necessary and sufficient conditions for A to be conservative or regular.

Theorem 1.1 *The matrix $A = (a_{nk})_{k=0}^{\infty}$ is conservative iff each of the following is true:*

(i). $\sup_n \sum_{k=0}^{\infty} |a_{nk}| < \infty$;

(ii). $\lim_{n\to\infty} a_{nk} = a_k$ *exists for all* $k \in \mathbf{N}$;

(iii). $\lim_{n\to\infty} \sum_{k=0}^{\infty} a_{nk} = a$ *exists.*

Under these circumstances $\sum a_k$ is absolutely convergent, and for $x = \{x_k\} \in c$,

$$\lim Ax = a \lim x + \sum_k a_k(x_k - \lim x) = (a - \sum a_k) \lim x + \sum a_k x_k.$$

In particular, A is regular iff (i), (ii) and (iii) hold with $a_k = 0$ for all k and with $a = 1$.

Much of this chapter is concerned with the question of determining whether a given matrix $A = (a_{nk}) \in B(l_p)$. There are two ways that one might approach this question. One way is to try, for a general matrix, to determine conditions, expressed in terms of the entries of the matrix, that are either necessary or sufficient for A to belong to $B(l_p)$. This has proved very difficult. Even where success has been achieved, notably in the case $p = 2$ [23], the conditions are unwieldy. Another approach is to consider some restricted class of matrices and to try to determine necessary and sufficient conditions for the members of that class to belong to $B(l_p)$. This has been the approach of [6] and [21]. Conditions of a simple kind that are applicable to, for example, Nörlund and weighted mean matrices, have been obtained by D. Borwein, F. P. Cass and W. Kratz. It is this latter approach that we take in sections two and three of this chapter. In chapter two we will discuss some necessary and sufficient conditions for a general non-negative matrix to belong to $B(l_p)$. This lies somewhere between the two approaches.

1.1.1 Some matrices and their relationships

Definition 1.2 *Let $a = \{a_n\}_{n=0}^{\infty}$ be a sequence of non-negative numbers, such that $A_n = a_0 + a_1 + \ldots + a_n > 0$. The* **Nörlund matrix** *$N_a = (a_{nk})$ is defined by*

$$a_{nk} = \begin{cases} a_{n-k}/A_n & \text{for } 0 \leqslant k \leqslant n, \\ 0 & \text{for } k > n. \end{cases}$$

The condition $a_n/A_n \to 0$ is necessary and sufficient for the regularity of N_a.

Definition 1.3 *Let $a = \{a_n\}_{n=0}^{\infty}$ be a sequence of non-negative numbers, such that $A_n = a_0 + a_1 + \ldots + a_n > 0$. The* **weighted mean matrix** *$M_a = (a_{nk})$ is defined by*

$$a_{nk} = \begin{cases} a_k/A_n & \text{for } 0 \leqslant k \leqslant n, \\ 0 & \text{for } k > n. \end{cases}$$

The condition $A_n \to \infty$ is necessary and sufficient for the regularity of M_a.

Definition 1.4 *Suppose*

$$\mu = \begin{pmatrix} \mu_0 & 0 & 0 & \cdots \\ 0 & \mu_1 & 0 & \cdots \\ 0 & 0 & \mu_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is any diagonal matrix. The **Hausdorff matrix** *defined by μ is $H(\mu) = \delta\mu\delta$, where $\delta = (d_{nk})$ is the lower triangular matrix defined by*

$$d_{nk} = \begin{cases} (-1)^k \begin{pmatrix} n \\ k \end{pmatrix} & \textit{for } 0 \leqslant k \leqslant n, \quad \textit{and,} \\ 0 & \textit{for } k > n. \end{cases} \tag{1.2}$$

It is readily checked that $\delta^2 = I$, the identity matrix. Hence given two sequences s and t, We have $s = \delta t$ iff $t = \delta s$. (For more details see [29, 11.1].)

If we write $\Delta^0 \mu_n = \mu_n$ and $\Delta^m \mu_n = \Delta^{m-1}\mu_n - \Delta^{m-1}\mu_{n+1}$, for $m \in \mathbf{N}$, then the entries of $H(\mu) = (\lambda_{nk})$ are given by

$$\lambda_{nk} = \begin{cases} \begin{pmatrix} n \\ k \end{pmatrix} \Delta^{n-k}\mu_k & \text{for } 0 \leqslant k \leqslant n, \text{ and,} \\ 0 & \text{for } k > n. \end{cases} \tag{1.3}$$

The proof of the following theorem can be found in [29, §11.9].

Theorem 1.2 (Hausdorff) *In order that the Hausdorff transformation $H(\mu) = \delta\mu\delta$ be regular, it is necessary and sufficient that $\mu_k = \int_0^1 t^k \, d\alpha(t)$ for some real-valued function α on $[0, 1]$ satisfying*

(i). *α is of bounded variation.*

(ii). $\alpha(0) = \alpha(+0) = 0, \alpha(1) = 1.$

The following are examples of Hausdorff matrices. All are regular.

(i). The Euler matrix (E, q), $q > 0$, given by $\mu_n = (q+1)^{-n}$;

(ii). Hõlder matrices (H, k), $k \geqslant 0$, given by $\mu_n = (n+1)^{-k}$;

(iii). Cesáro matrices (C, k), $k \geqslant 0$, given by $\mu_n = \begin{pmatrix} n+k \\ k \end{pmatrix}^{-1}$.

One property that characterizes the class of Hausdorff matrices is commutativity. We have

Theorem 1.3 *Any two Hausdorff matrices commute. The class of all Hausdorff matrices consists of all those matrices that commute with the Cesàro matrix $(C, 1)$ (or any Hausdorff matrix $H(\mu)$ all of whose μ_n differ).*

A proof of Theorem(1.3) can be found in [29, 11.3].

Suppose $f(x)$ is a function defined at least for $x = x_0, x_1, \ldots, x_m$. If all the x_k's are different, following [34], we define the **divided differences** inductively by

$$[x_0] = f(x_0);$$

$$[x_0, x_1] = \frac{[x_0] - [x_1]}{x_0 - x_1}$$

$$[x_0, x_1, x_2] = \frac{[x_0, x_1] - [x_1, x_2]}{x_0 - x_2}$$

$$\cdots\cdots$$

$$[x_0, x_1, \ldots, x_m] = \frac{[x_0, \ldots, x_{m-1}] - [x_1, \ldots, x_m]}{x_0 - x_m}.$$

We can deduce by induction that

$$[x_0, x_1, \ldots, x_m] = \sum_{k=0}^{m} \frac{f(x_k)}{\prod_{j \neq k}(x_k - x_j)} \tag{1.4}$$

Now suppose the function f is analytic on a domain $G \subset \mathbf{C}$, which contains the points $x_0, x_1, x_2, \ldots$. Let $C = C_m$ be a Jordan contour in G which encloses $x_0, x_1, \ldots, x_m$. Then

$$[x_0, x_1, \ldots, x_m] = \frac{(-1)^{m+1}}{2\pi i} \int_C \frac{f(z)\,dz}{(x_0 - z) \cdots (x_m - z)}$$

$$= \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - x_0) \cdots (z - x_m)} \tag{1.5}$$

The right side of (1.5) exists, however, even if some of the x_k's coincide, whereas the right hand side of (1.4) makes no sense. So for an analytic function f, we can define divided differences by (1.5).

Let $\lambda_0 \geqslant 0, \lambda_k > 0$ for $k \in \mathbf{N}$, f and G be as above. We write

$$\xi_{nk} = (-1)^{n-k} \lambda_{k+1} \cdots \lambda_n [\lambda_k, \ldots, \lambda_n]$$

$$= -\lambda_{k+1}\cdots\lambda_n \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(\lambda_k - z)\cdots(\lambda_n - z)} \quad \text{for } 0 \leqslant k \leqslant n; \tag{1.6}$$

$\xi_{nk} = 0 \quad$ for $k > n$.

If $\lambda_n = n$ in (1.6), we can prove inductively that

$$[\lambda_k, \ldots, \lambda_n] = \frac{(-1)^{n-k}}{(n-k)!}\Delta^{n-k} f(\lambda_k)$$

so that for $0 \leqslant k \leqslant n$, (1.6) becomes

$$\xi_{nk} = (-1)^{n-k}\lambda_{k+1}\cdots\lambda_n[\lambda_k, \ldots, \lambda_n]$$

$$= \frac{n!}{k!}\frac{\Delta^{n-k} f(\lambda_k)}{(n-k)!} = \binom{n}{k}\Delta^{n-k} f(\lambda_k) = \binom{n}{k}\Delta^{n-k} f(k).$$

Comparing this with (1.3), we see that $\Lambda = (\xi_{nk})$ is the Hausdorff matrix $H(\mu)$ where

$$\mu = \begin{pmatrix} f(0) & 0 & 0 & \cdots \\ 0 & f(1) & 0 & \cdots \\ 0 & 0 & f(2) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Thus we see that, if there is an appropriate analytic function f such that $f(n) = \mu_n$, the entries of a Hausdorff matrix $H(\mu)$ can be given by (1.6) by putting $\lambda_n = n$.

Definition 1.5 *Suppose that $\lambda = \{\lambda_n\}_{n=0}^{\infty}$ is a sequence of real numbers with $\lambda_0 \geqslant 0, \lambda_n > 0$ for $n = 1, 2, \ldots$. For $0 \leqslant k \leqslant n$, let*

$$\lambda_{nk}(t) = -\lambda_{k+1}\cdots\lambda_n \frac{1}{2\pi i}\int_C \frac{t^z\,dz}{(\lambda_k - z)\cdots(\lambda_n - z)}, \quad 0 < t \leqslant 1,$$

and (1.7)

$$\lambda_{nk}(0) = \lambda_{nk}(0+),$$

where $C = C_{nk}$ is a positively sensed Jordan contour enclosing $\lambda_k, \lambda_{k+1}, \ldots, \lambda_n$. Here and elsewhere we adopt the convention that all empty products, like $\lambda_{k+1}\cdots\lambda_n$ with $k = n$, have the value 1. Let α be a function of bounded variation on $[0, 1]$ and define

$$\lambda_{nk} = \int_0^1 \lambda_{nk}(t)\, d\alpha(t) \quad \text{for } 0 \leqslant k \leqslant n, \qquad \lambda_{nk} = 0 \quad \text{for } k > n, \tag{1.8}$$

and denote the triangular matrix $(\lambda_{nk})_{n,k=0}^{\infty}$ *by* $H(\lambda, \alpha)$. *This matrix is called a* **generalized Hausdorff matrix.**

Remark. If $H(\mu)$ is a regular Hausdorff matrix, then Theorem(1.2) shows that $\mu_k = \int_0^1 t^k\, d\alpha(t)$ where α is a real function on $[0,1]$ satisfying (i) and (ii) of Theorem(1.2). Hence by taking

$$f(z) = \int_0^1 t^z\, d\alpha(t), \tag{1.9}$$

the Mellin transform of α, we see that $H(\mu)$ is also a generalized Hausdorff matrix in the sense of (1.7) and (1.8).

Remark. In some references, generalized Hausdorff matrices were defined in more general senses. In [7], for example, the definition is the following.

> Let Ω be a simply connected region that contains every positive λ_n and suppose that , for $n = 0, 1, 2, \ldots$, Γ_n is a positively sensed Jordan contour lying in Ω and enclosing every $\lambda_k \in \Omega$ with $0 \leqslant k \leqslant n$. Suppose that f is analytic and that $f(\lambda_0)$ is defined even when $\lambda_0 \notin \Omega$. Define
>
> $$\lambda_{nk} = \begin{cases} -\dfrac{\lambda_{k+1} \cdots \lambda_n}{2\pi i} \displaystyle\int_{\Gamma_n} \frac{f(z)\, dz}{(\lambda_k - z) \cdots (\lambda_n - z)} + \delta_k & \text{for } 0 \leqslant k \leqslant n, \\ = 0 & \text{for } k > n, \end{cases}$$
>
> where $\delta_k = f(\lambda_0)$ if $k = 0$ and $\lambda_0 \notin \Omega$, and $\delta_k = 0$ otherwise.

A weighted mean matrix is a special case of a generalized Hausdorff matrix. This fact was originally proved by Hausdorff in [30] for the case where all the λ_n are distinct. A continuity argument can be used to show that it is also true without the λ_n being distinct. There is, however, a direct and short proof which we now give.

First we need some notation. Let $\{D_n\}$, $\{\lambda_n\}$ and $\{d_n\}$ be three sequences of real numbers, such that $d_n > 0$, $\lambda_n > -1$, for $n = 0, 1, \ldots$, and suppose they are related in the following way.

$$D_0 = (1+\lambda_0)d_0 = 1$$

$$D_n = (1+\lambda_n)d_n = \left(1+\frac{1}{\lambda_1}\right)\cdots\left(1+\frac{1}{\lambda_n}\right) \quad \text{for } n \geqslant 1.$$

The above equations imply that

$$D_n = \lambda_{n+1}d_{n+1} = \frac{\lambda_0}{1+\lambda_0} + \sum_{k=0}^{n} d_k \quad \text{for } n \geqslant 0. \tag{1.10}$$

Lemma 1.4 *Suppose $\{D_n\}$, $\{d_n\}$ and $\{\lambda_n\}$ are sequences of real numbers as described above. Then*

$$\int_0^1 \lambda_{nk}(t)\,dt = \frac{d_k}{D_n} \quad \textit{for } 0 \leqslant k \leqslant n.$$

PROOF: Let Γ be a circle enclosing $\lambda_k, \ldots, \lambda_n$ and lying in the half plane $\operatorname{Re} z > -\delta$ where $0 < \delta < 1$. For $0 < t \leqslant 1$ and $z \in \Gamma$, we have

$$\left|\frac{t^z}{(\lambda_k - z)\cdots(\lambda_n - z)}\right| \leqslant Mt^{-\delta}$$

for some positive M independent of t and z. Hence, by Fubini's theorem,

$$\int_0^1 \lambda_{nk}(t)\,dt = -\frac{\lambda_{k+1}\cdots\lambda_n}{2\pi i}\int_0^1 dt \int_\Gamma \frac{t^z\,dz}{(\lambda_k - z)\cdots(\lambda_n - z)}$$

$$= -\frac{\lambda_{k+1}\cdots\lambda_n}{2\pi i}\int_\Gamma \frac{dz}{(z+1)(\lambda_k - z)\cdots(\lambda_n - z)}.$$

Let

$$g(z) = \frac{1}{(z+1)(\lambda_k - z)\cdots(\lambda_n - z)}.$$

Then $\int_{|z|=m} g(z)\,dz \to 0$ as $m \to \infty$. Using the fact that, when m is sufficiently large,

$$\frac{1}{2\pi i}\int_{|z|=m} g(z)\,dz = \sum_{j=k}^{n} \operatorname{Res}(g(z),\lambda_j) + \operatorname{Res}(g(z),-1)$$

$$= \frac{1}{2\pi i}\int_{\Gamma} g(z)\,dz + \operatorname{Res}(g(z),-1),$$

we have

$$\frac{1}{2\pi i}\int_{\Gamma} g(z)\,dz = -\operatorname{Res}(g(z),-1) = -\frac{1}{(\lambda_k+1)\cdots(\lambda_n+1)}.$$

Consequently

$$\int_0^1 \lambda_{nk}(t)\,dt = \frac{\lambda_{k+1}\cdots\lambda_n}{(\lambda_k+1)\cdots(\lambda_n+1)} = \frac{d_k}{D_n}. \qquad \square$$

Remark. The proof given above is from [9]. A similar proof can be found in [8].

Given a sequence of non-negative numbers $\{a_n\}$ with $a_0 > 0$, we can define $d_n = a_n/a_0$ for $n \geqslant 0$ and $\{\lambda_n\}$ by (1.10). Lemma(1.4), together with equations (1.7) and (1.8) show that the weighted mean matrix M_a is $H(\lambda,\alpha)$ for the function $\alpha(t) = t$. Conversely if $\lambda_0 = 0$, then this same generalized Hausdorff matrix is a weighted mean matrix.

1.1.2 Some General Results

The sufficiency part of the next theorem will be used frequently in the following sections. That part was independently proved and used by different authors, for example, D. Borwein, A. Jakimovsky in [11] and Vere-Jones in [46]. We give the proof for ease of reference. The necessity part of the theorem will be proved in §2.3, where it is a corollary of a more general result.

Theorem 1.5 *Let $p > 1$. Suppose $a_{nk} \geqslant 0$, for $n,k = 0,1,2,\ldots$. Then for $A = (a_{nk})_{n,k=0}^{\infty}$ to be in $B(l_p)$, it is both necessary and sufficient that there exists a sequence $\{b_n\}$ of positive numbers, such that*

$$\sup_{n\geqslant 0}\sum_{k=0}^{\infty} a_{nk}\left(\frac{b_k}{b_n}\right)^{1/p} = M_1 < \infty$$

$$\sup_{k\geqslant 0}\sum_{n=0}^{\infty} a_{nk}\left(\frac{b_n}{b_k}\right)^{1/p'} = M_2 < \infty. \tag{1.11}$$

In this case $\|A\|_p \leqslant {M_1}^{1/p'}{M_2}^{1/p}$.

PROOF OF SUFFIC.ENCY: Let $y_n = \sum_{k=0}^{\infty} a_{nk}x_k$. Then by Hõlder's inequality,

$$|y_n|^p \leqslant \left[\sum_{k=0}^{\infty} a_{nk}\left(\frac{b_k}{b_n}\right)^{1/p}\right]^{p-1} \sum_{k=0}^{\infty} a_{nk}\left(\frac{b_n}{b_k}\right)^{1/p'} |x_k|^p$$

$$\leqslant M_1^{p-1}\sum_{k=0}^{\infty} a_{nk}\left(\frac{b_n}{b_k}\right)^{1/p'} |x_k|^p.$$

Hence

$$\sum_{n=0}^{\infty}|y_n|^p \leqslant M_1^{p-1}\sum_{k=0}^{\infty}|x_k|^p\sum_{n=0}^{\infty} a_{nk}\left(\frac{b_n}{b_k}\right)^{1/p'}$$

$$\leqslant M_1^{p-1}M_2\textstyle\sum_{k=0}^{\infty}|x_k|^p.$$ □

The next lemma is Theorem 4 of [11] with superficial changes to make it a little easier to apply.

Lemma 1.6 *Suppose that* $a_{nk} \geqslant 0$ *for* $0 \leqslant k, n < \infty$, *and that* $\{b_n\}$ *is a bounded sequence of positive numbers such that* $\sum b_n = \infty$. *Let*

$$\sigma_n = \sum_{k=0}^{n} a_{nk}\left(\frac{b_k}{b_n}\right)^{1/p}.$$

If $\sigma = \liminf \sigma_n$ *and* $A = (a_{nk})$, *then* $\|A\|_p \geqslant \sigma$ *for* $p \geqslant 1$. *In particular, if* $\sigma = \infty$, *then* $A \notin B(l_p)$.

PROOF: Suppose without loss of generality that $\sigma > 0$, and let $0 < \mu < \lambda < \sigma$. Let

$$T_n = \prod_{k=0}^{n}\left(1 - \frac{b_k}{b}\right)^{-1} \quad \text{where} \quad b > \sup_{k\geqslant 0} b_k,$$

and $t_0 = T_0$. Then $t_n = T_n - T_{n-1} = T_n b_n / b$ for $n \geqslant 1$, and $T_n = t_0 + t_1 + \cdots + t_n \to \infty$ as $n \to \infty$. Let

$$y_n = \sum_{k=0}^{n} a_{nk} x_k \quad \text{where} \quad x_k = \left(\frac{b_k}{T_k^\epsilon}\right)^{1/p}, \quad \epsilon > 0.$$

We observe that $x = \{x_n\} \in l_p$ because $\sum_n x_n^p = b \sum_n (t_n / T_n^{1+\epsilon}) < \infty$ by Dini's theorem. There is a positive integer N independent of ϵ such that for $n \geqslant N$

$$\begin{aligned} y_n &= x_n \sum_{k=0}^{n} a_{nk} \left(\frac{b_k}{b_n}\right)^{1/p} \left(\frac{T_n}{T_k}\right)^{\epsilon/p} \\ &\geqslant x_n \sum_{k=0}^{n} a_{nk} \left(\frac{b_k}{b_n}\right)^{1/p} \geqslant \lambda x_n. \end{aligned}$$

Now again by Dini's theorem, $\sum_{n=0}^{\infty} (b t_n / T_n) = \infty$ but $x_n^p \to b t_n / T_n$ for all n as $\epsilon \to 0$, so for any fixed N, $(\sum_{n=N}^{\infty} x_n^p) / (\sum_{n=0}^{\infty} x_n^p) \to 1$ as $\epsilon \to 0$. Thus we can choose ϵ so small that

$$\sum_{n=N}^{\infty} x_n^p = b \sum_{n=N}^{\infty} \frac{t_n}{T_n^{1+\epsilon}} \geqslant \left(\frac{\mu}{\lambda}\right)^p \sum_{n=0}^{\infty} x_n^p.$$

Then

$$\sum_{n=0}^{\infty} y_n^p \geqslant \lambda^p \sum_{n=N}^{\infty} x_n^p \geqslant \mu^p \sum_{n=0}^{\infty} x_n^p.$$

Therefore $\|A\|_p \geqslant \mu$ and, since μ is an arbitrary number in the interval $(0, \sigma)$, it follows that $\|A\|_p \geqslant \sigma$. □

1.2 Nörlund matrices as operators on l_p

Given a Nörlund matrix N_a defined by the sequence $a = \{a_k\}$, $a_0 > 0$, $a_k \geqslant 0$ for $k \in \mathbf{N}$, we first consider the case where $\sum_{k=0}^{\infty} a_k$ is finite. The following theorem is similar to part of Theorem 1 in [6]. The differences are that, in [6], complex sequences were considered and the matrix $A = (a_{nk})$ was given by $a_{nk} = a_{n-k}$ for $0 \leqslant k \leqslant n$ instead of by $a_{nk} = a_{n-k}/A_n$. This does not change the transformation in any essential way.

Theorem 1.7 (D. Borwein and F. P. Cass[6]) *Let* $1 \leqslant p < \infty$, *and* $\|a\|_1 = \sum_{k=0}^{\infty} a_k < \infty$. *Then* $N_a \in B(l_p)$ *and* $\|N_a\|_p \leqslant \|a\|_1/a_0$.

PROOF: For $1 \leqslant p < \infty$, $x = \{x_k\} \in l_p$, Hölder's inequality gives us:

$$|y_n|^p \leqslant \left(\frac{1}{a_0}\sum_{k=0}^{n} a_{n-k}|x_k|\right)^p = \frac{1}{a_0{}^p}\left(\sum_{k=0}^{n} a_{n-k}{}^{1/p}|x_k|a_{n-k}{}^{1/p'}\right)^p$$

$$\leqslant \frac{1}{a_0{}^p}\left(\sum_{k=0}^{n} a_{n-k}|x_k|^p\right)\|a\|_1^{p-1}.$$

Hence

$$\sum_{n=0}^{\infty}|y_n|^p \leqslant \frac{\|a\|_1^{p-1}}{a_0{}^p}\sum_{n=0}^{\infty}\sum_{k=0}^{n} a_{n-k}|x_k|^p = \frac{\|a\|_1^p}{a_0{}^p}\|x\|_p^p. \qquad \square$$

The next theorem, from [6], gives a sufficient condition for $N_a \in B(l_p)$.

Theorem 1.8 (D. Borwein and F. P. Cass) *For* $1 < p < \infty$, *suppose that*

$$(n+1)a_n \leqslant HA_n, \quad \text{for some positive number } H, \tag{1.12}$$

then $N_a \in B(l_p)$, *and*

$$\|N_a\| \leqslant 2^{1/pp'}(p'H+1) \leqslant 2^{1/4}(p'H+1).$$

The condition given in the above theorem, however, is not necessary in general. D. Borwein gave an example in [5] with the sequence $\{na_n/A_n\}$ unbounded, but for which $N_a \in B(l_p)$. On the other hand, F. P. Cass and W. Kratz proved in [21], *inter alia*, that if

$$a_n = f(n), \quad \text{for some logarithmico-exponential function } f, \tag{1.13}$$

then (1.12) is both necessary and sufficient for $N_a \in B(l_p)$. Condition (1.13) implies that the sequence $\{a_n/A_n\}$ is eventually monotonic. Cass and Kratz also showed in

Lemma 3 of [21] that $\{na_n/A_n\}$ always has a finite or infinite limit when (1.13) is true. We show next that the sequence $\{a_n\}$ need not be associated with a logarithmico-exponential function and that all that is required is the existence of $\lim_n na_n/A_n$.

Theorem 1.9 *Let $1 \leqslant p < \infty$ and suppose $na_n/A_n \to \alpha$ as $n \to \infty$. Then $N_a \in B(l_p)$ iff $\alpha < \infty$.*

To prove the above theorem, we need the following lemma.

Lemma 1.10 *Suppose that $a_n > 0$, and $\lim_n \dfrac{a_n}{A_n} = 0$. Let $h_n = \max_{k \leqslant n}[A_k/a_k]$, where $[A_k/a_k]$ stands for the largest integer less than or equal to A_k/a_k. Then*

$$A_n \geqslant \sum_{k=n-h_n}^{n} a_k \geqslant \left(1 - \frac{1}{e}\right) A_n. \tag{1.14}$$

Further if $na_n/A_n \to \infty$, then $h_n = o(n)$.

Proof: If $h_n = n$ then (1.14) is trivial. Suppose, therefore, that $n - h_n \geqslant 1$. Then

$$\log \frac{A_n}{A_{n-h_n-1}} = \int_{A_{n-h_n-1}}^{A_n} \frac{dx}{x} = \sum_{k=n-h_n}^{n} \int_{A_{k-1}}^{A_k} \frac{dx}{x}$$

$$\geqslant \sum_{k=n-h_n}^{n} \frac{a_k}{A_k} \geqslant (h_n + 1) \min_{k \leqslant n} \frac{a_k}{A_k} \geqslant 1,$$

and so $A_{n-h_n-1}/A_n \leqslant e^{-1}$. Thus

$$\frac{1}{A_n} \sum_{k=n-h_n}^{n} a_k = 1 - \frac{A_{n-h_n-1}}{A_n} \geqslant 1 - \frac{1}{e}.$$

Now suppose that $na_n/A_n \to \infty$. Let $m \geqslant 1$ be a fix integer and let $n \geqslant m$. Then

$$\frac{h_n}{n} \leqslant \frac{1}{n} \max_{k \leqslant n} \frac{A_k}{a_k} \leqslant \frac{1}{n} \max_{k<m} \frac{A_k}{a_k} + \frac{1}{n} \max_{m \leqslant k \leqslant n} \frac{A_k}{a_k}$$

$$= \frac{1}{n} \max_{k<m} \frac{A_k}{a_k} + \max_{m \leqslant k \leqslant n} \frac{kA_k}{nka_k} \leqslant \frac{1}{n} \max_{k<m} \frac{A_k}{a_k} + \max_{m \leqslant k \leqslant n} \frac{A_k}{ka_k}.$$

So we have

$$0 \leqslant \limsup_{n\to\infty} \frac{h_n}{n} \leqslant \lim_{n\to\infty} \frac{1}{n} \max_{k<m} \frac{A_k}{a_k} + \limsup_{n\to\infty} \max_{m\leqslant k\leqslant n} \frac{A_k}{ka_k}$$

$$\leqslant \max_{k\geqslant m} \frac{A_k}{ka_k}.$$

Now let $m \to \infty$, it follows that $h_n/n \to 0$. □

Theorem(1.9) and Lemma(1.10) were originally proved by D. Borwein with the extra condition that $\{a_n/A_n\}$ be eventually monotonic. X. Gao showed that this monotonicity condition is not needed. See [5].

PROOF OF THEOREM(1.9): That $N_a \in B(l_p)$ when $\alpha < \infty$ follows from Theorem-(1.8). Now we only need to show that $N_a \notin B(l_p)$ when $\alpha = \infty$. If $a_n/A_n \not\to 0$, then $\sum_n |a_n/A_n|^p = \infty$ for all p, so that $N_a e_0 \notin l_p$. So $N_a \notin B(l_p)$. Hence we suppose that $a_n/A_n \to 0$ as $n \to \infty$, and, without loss of generality, we suppose that all a_n are positive.

Let $\delta = 1/p$. With a view to applying Lemma(1.6) with $b_n = 1/(n+1)$ and using Lemma(1.10), we have

$$\sigma_n = \frac{(n+1)^\delta}{A_n} \sum_{k=0}^{n} a_k (n+1-k)^{-\delta} \geqslant \frac{(n+1)^\delta}{A_n} \sum_{k=n-h_n}^{n} a_k (n+1-k)^{-\delta}$$

$$\geqslant \left(\frac{n+1}{h_n+1}\right)^\delta \frac{1}{A_n} \sum_{k=n-h_n}^{n} a_k \geqslant \left(1 - \frac{1}{e}\right)\left(\frac{n+1}{h_n+1}\right)^\delta \to \infty.$$

It follows, from Lemma(1.6) that $N_a \notin B(l_p)$. □

Remark. It was also proved by D. Borwein in [5] that if, in addition to the existence of $\alpha = \lim na_n/A_n$, the sequence $\{n^c a_n\}$ is eventually monotonic for every constant $c \neq 1 - \alpha$, then

$$\lim_{n\to\infty} \sigma_n = \frac{\Gamma(\alpha+1)\Gamma(1/p')}{\Gamma(\alpha+1/p')} \leqslant \|N_a\|_p. \tag{1.15}$$

Further, the monotonicity condition is not needed when $\alpha = 0$.

If the sequence $\{a_n\}_{n=0}^{\infty}$ is given by $a_n = \binom{n+\alpha-1}{\alpha-1}$ for some $\alpha > 0$, and hence $A_n = \binom{n+\alpha}{\alpha}$, then the Nörlund matrix N_a is the Cesáro matrix (C,α). We can check that the limit of na_n/A_n in this case is α. G. H. Hardy showed in [28] that for $p > 1$, $\|(C,\alpha)\|_p = \Gamma(\alpha+1)\Gamma(1/p')/\Gamma(\alpha+1/p')$. So in this case equality is attained in (1.15).

1.3 Generalized Hausdorff and Weighted Mean Matrices on l_p

Consider a generalized Hausdorff matrix (λ_{nk}) as in Definition(1.5) Using the residue theorem, we have for $0 < t \leqslant 1$,

$$\lambda_{nk}(t) = -\lambda_{k+1}\cdots\lambda_n \frac{1}{2\pi i}\int_C \frac{t^z\,dz}{(\lambda_k - z)\cdots(\lambda_n - z)}$$

$$= -\lambda_{k+1}\cdots\lambda_n \sum_{\substack{k\leqslant i\leqslant n\\ \lambda_i \text{ different}}} \operatorname{Res}\left(\frac{t^z}{(\lambda_k - z)\cdots(\lambda_n - z)}, \lambda_i\right)$$

$$= -\lambda_{k+1}\cdots\lambda_n \sum_{\substack{k\leqslant i\leqslant n\\ \lambda_i \text{different}}} \operatorname{Res}\left(t^z\left[\prod_{\substack{k\leqslant j\leqslant n\\ \lambda_j \text{different}}} (\lambda_j - z)^{r_j}\right]^{-1}, \lambda_i\right)$$

$$= -\lambda_{k+1}\cdots\lambda_n \sum_{\substack{k\leqslant i\leqslant n\\ \lambda_i \text{different}}} \lim_{z\to\lambda_i} \frac{1}{(r_i-1)!}\frac{d^{r_i-1}}{dz^{r_i-1}}\left\{(z-\lambda_i)^{r_i}\left[t^z\left(\prod_{\substack{k\leqslant j\leqslant n\\ \lambda_j \text{different}}}(\lambda_j - z)^{r_j}\right)^{-1}\right]\right\}$$

$$= -\lambda_{k+1}\cdots\lambda_n \sum_{\substack{k\leqslant i\leqslant n\\ \lambda_i \text{different}}} \lim_{z\to\lambda_i} \frac{1}{(r_i-1)!}\frac{d^{r_i-1}}{dz^{r_i-1}}\left\{(-1)^{r_i}t^z\left(\prod_{\substack{k\leqslant j\leqslant n\\ \lambda_j\neq\lambda_i}} (\lambda_j - z)\right)^{-1}\right\}$$

$$= \sum_{\substack{k\leqslant i\leqslant n\\ \lambda_i \text{different}}} \lim_{z\to\lambda_i} \sum_{m=0}^{r_i-1} f_{im}(z)t^z(\log t)^m = \sum_{\substack{k\leqslant i\leqslant n\\ \lambda_i \text{different}}} \sum_{m=0}^{r_i-1} f_{im}(\lambda_i)t^{\lambda_i}(\log t)^m, \tag{1.16}$$

where for each i and m satisfying $k \leqslant i \leqslant n$ and $0 \leqslant m < r_i$, f_{im} is a function analytic at λ_i.

Note that for $i > 0$, because $\lambda_i > 0$,

$$\lim_{t\to 0^+} f_{im}(\lambda_i)t^{\lambda_i}(\log t)^m = 0, \text{ for } m = 0, 1, 2, \ldots. \tag{1.17}$$

So $\lambda_{nk}(0) = \lambda_{nk}(0+) = \lim_{t\to 0^+} \lambda_{nk}(t) = \sum\sum 0 = 0$ when $k > 0$. Even when $k = 0$, if $\lambda_0 > 0$, we also have $\lambda_{n0}(0) = 0$ for the same reason.

If $\lambda_0 = 0$, λ_0 is different from all other λ_j, so $r_0 = 1$. By (1.16) and (1.17) we have

$$\lim_{t\to 0^+} \lambda_{n0}(t) = \lim_{t\to 0^+} \lim_{z\to\lambda_0=0} (-\lambda_1\cdots\lambda_n)(-1)t^z \left(\prod_{0<j\leqslant n}(\lambda_j - z)\right)^{-1}$$

$$= \lim_{t\to 0^+} \lambda_1\cdots\lambda_n \frac{1}{(\lambda_1 - 0)\cdots(\lambda_n - 0)}$$

$$= 1$$

In summary, we have

$$\lambda_{nk}(0) = \begin{cases} 1, & \text{if } k = 0 \text{ and } \lambda_0 = 0; \\ 0, & \text{otherwise.} \end{cases} \tag{1.18}$$

Next we prove that $\lambda_{nk}(t) \geqslant 0$ for all n and k. Following [34], take any complex number z, any real number y, $z \neq y$, and any sequence $\{x_k\}$ of real numbers, such that $z \neq x_k$ for all k. For $n = 0, 1, 2, \ldots$, we have

$$\frac{1}{z-y} = \frac{1}{z-x_0} + \frac{y-x_0}{z-x_0}\frac{1}{z-y}$$

$$= \frac{1}{z-x_0} + \frac{y-x_0}{(z-x_0)(z-x_1)} + \frac{(y-x_0)(y-x_1)}{(z-x_0)(z-x_1)}\frac{1}{z-y}$$

$$= \frac{1}{z-x_0} + \frac{y-x_0}{(z-x_0)(z-x_1)} + \cdots + \frac{(y-x_0)\cdots(y-x_{i-1})}{(z-x_0)\cdots(z-x_i)} + \cdots$$

$$+ \frac{(y-x_0)\cdots(y-x_{n-1})}{(z-x_0)\cdots(z-x_n)} + \frac{(y-x_0)\cdots(y-x_n)}{(z-x_0)\cdots(z-x_n)}\frac{1}{z-y} \tag{1.19}$$

Multiplying by $t^z/2\pi i$ and integrating over C, we have for y inside C,

$$t^y = \sum_{k=0}^{n}[x_0,\ldots,x_k](y-x_0)\cdots(y-x_{k-1}) + [x_0,\ldots,x_n,y](y-x_0)\cdots(y-x_n)$$

$$= P_n(y) + [x_0,\ldots,x_n,y](y-x_0)\cdots(y-x_n).$$

where $[x_0,\ldots,x_m]$ is given by (1.5) with $f(z)=t^z$ for $m=0,1,2,\ldots$.

Suppose $[a,b]$ is an interval containing all $x_0,x_1,\ldots,x_n$, we see from the above that $g(y)=t^y-P_n(y)$ has $n+1$ zeros on $[a,b]$. By Rolle's theorem, $g^{(n)}(\xi)=0$ for some $\xi\in[a,b]$, that is

$$[x_0,\ldots,x_n] = \frac{1}{n!}t^{\xi}(\log t)^n.$$

So

$$\lambda_{nk}(t) = (-1)^{n-k}\lambda_{k+1}\cdots\lambda_n[\lambda_k,\ldots,\lambda_n] = (-1)^{n-k}\lambda_{k+1}\cdots\lambda_n t^{\xi}(\log t)^{n-k} \geqslant 0 \quad (1.20)$$

The following two lemmas on the properties of generalized Hausdorff matrices will be needed to establish the main theorems of this section. Both of these lemmas are special cases of results of Borwein, Cass and Sayre in [7]. They were basic in establishing necessary and sufficient conditions for the regularity of generalized Hausdorff matrices.

Lemma 1.11 *For $t>0$ and the matrix $(\lambda_{nk}(t))$ defined by* (1.7)*, the limit, $l_k(t)=\lim_{n\to\infty}\lambda_{nk}(t)$, exists for $k=0,1,2,\ldots$. If, in addition, $\sum_{n=1}^{\infty}1/\lambda_n=\infty$, then $l_k(t)=0$ for $k=0,1,2,\ldots$.*

PROOF: Suppose $t>0$. For $0\leqslant k\leqslant n$,

$$\lambda_{nk}(t) - \lambda_{n+1,k}(t)$$

$$= -\lambda_{k+1}\cdots\lambda_n\frac{1}{2\pi i}\int_C\left[\frac{1}{(\lambda_k-z)\cdots(\lambda_n-z)} - \frac{\lambda_{n+1}}{(\lambda_k-z)\cdots(\lambda_{n+1}-z)}\right]t^z\,dz$$

$$= -\lambda_{k+1}\cdots\lambda_n\frac{1}{2\pi i}\int_C\frac{-z}{(\lambda_k - z)\cdots(\lambda_{n+1} - z)}t^z\,dz$$

$$= -\lambda_{k+1}\cdots\lambda_n\frac{1}{2\pi i}\int_C\frac{\lambda_k - z - \lambda_k}{(\lambda_k - z)\cdots(\lambda_{n+1} - z)}t^z\,dz$$

$$= [\lambda_{k+1}\lambda_{n+1,k+1}(t) - \lambda_k\lambda_{n+1,k}(t)]/\lambda_{n+1} \tag{1.21}$$

We define $\Lambda_{nk}(t) = \sum_{i=0}^{k}\lambda_{ni}(t)$, then it follows that

$$\Lambda_{nk}(t) - \Lambda_{n+1,k}(t) = [\lambda_{k+1}\lambda_{n+1,k+1}(t) - \lambda_0\lambda_{n+1,0}(t)]/\lambda_{n+1}. \tag{1.22}$$

First suppose that $\lambda_0 = 0$, then (1.22) and (1.20) give us $\Lambda_{nk}(t) - \Lambda_{n+1,k}(t) \geqslant 0$, hence $L_k(t) = \lim_{n\to\infty}\Lambda_{nk}(t)$ exists and hence so does $l_k(t) = L_k(t) - L_{k-1}(t) = \lim_{n\to\infty}\lambda_{nk}(t) \geqslant 0$. (Here we write $L_{-1}(t) = 0$.) Now (1.22) also shows that the series $\sum_{n=0}^{\infty}\lambda_{n+1,k+1}(t)/\lambda_{n+1}$ is convergent for all k when $\lambda_0 = 0$. Therefore $l_k(t) = 0$ for $k = 1, 2, \ldots$ when $\sum_{n=1}^{\infty} 1, \lambda_n$ is divergent.

Next we prove that $l_0(t) = 0$ when $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$ and $\lambda_0 = 0$. In this case, for sufficiently small $\eta > 0$,

$$\lambda_{n0}(t) = -\lambda_1\lambda_2\cdots\lambda_n\frac{1}{2\pi i}\int_C\frac{t^z\,dz}{(-z)(\lambda_1 - z)\cdots(\lambda_n - z)}$$

$$= -(\lambda_1 + \eta)(\lambda_2 + \eta)\cdots(\lambda_n + \eta)\frac{\gamma_n}{2\pi i}\int_{\tilde{C}}\frac{t^{z_1-\eta}\,dz_1}{(\eta - z_1)(\lambda_1 + \eta - z_1)\cdots(\lambda_n + \eta - z_1)},$$

where $z_1 = z + \eta$, $\tilde{C} = C + \eta$ and

$$0 \leqslant \gamma_n = \lambda_1\lambda_2\cdots\lambda_n/(\lambda_1 + \eta)\cdots(\lambda_n + \eta) \leqslant 1.$$

Since $\sum_{n=0}^{\infty} 1/(\lambda_n + \eta) = \infty$, it follows that $l_0(t) = 0$.

Finally suppose that $\lambda_0 > 0$. Let $\tilde{\lambda}_0 = 0$ and $\tilde{\lambda}_n = \lambda_{n-1}$ for $n \in \mathbf{N}$, and define $\tilde{\lambda}_{nk}(t)$ by (1.7) with $\tilde{\lambda}_n$ in place of λ_n, then $\lambda_{nk}(t) = \tilde{\lambda}_{n+1,k+1}(t)$ for $0 \leqslant k \leqslant n$, and hence $\lim_{n\to\infty}\lambda_{nk}(t) = l_k(t)$ exists for $k = 0, 1, 2, \ldots$. If we have, in addition, $\sum_n 1/\lambda_n = \infty$, then

$$l_k(t) = \lim_{n\to\infty} \lambda_{nk}(t) = \lim_{n\to\infty} \tilde{\lambda}_{n+1,k+1}(t) = \tilde{l}_{k+1}(t) = 0$$

for $k = 0, 1, 2, \ldots$ □

The next lemma was proved but not explicitly stated in [7].

Lemma 1.12 *For the matrix $(\lambda_{nk}(t))$ defined by* (1.7), *we have $\sum_{k=0}^{n} \lambda_{nk}(t) \leqslant 1$ for $0 \leqslant t \leqslant 1$. If, in addition. $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$, then*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \lambda_{nk}(t) = \begin{cases} 0, & \text{if } t = 0 \text{ and } \lambda_0 > 0, \\ 1, & \text{otherwise.} \end{cases}$$

PROOF: If $t = 0$, the result follows from (1.18). Assume $t > 0$.

First suppose $\lambda_0 = 0$. Take $y = \lambda_0 = 0, x_0 = \lambda_n, \ldots, x_n = \lambda_0$ in (1.19), we get

$$\frac{1}{z} = \frac{1}{z - \lambda_n} - \frac{\lambda_n}{(z - \lambda_{n-1})(z - \lambda_n)} + \cdots + (-1)^n \frac{\lambda_1 \cdots \lambda_n}{(z - \lambda_0) \cdots (z - \lambda_n)}.$$

Multiplying both sides by $t^z/2\pi i$ and integrating on C, we get

$$1 = t^0 = \frac{1}{2\pi i} \int_C \frac{t^z\,dz}{z} = \sum_{k=0}^{n} \lambda_{nk}(t).$$

Now for $\lambda_0 > 0$, let $\tilde{\lambda}_0 = 0$, $\tilde{\lambda}_n = \lambda_{n-1}$ for $n \in \mathbf{N}$ just as in the proof of the previous lemma, we get

$$\sum_{k=0}^{n} \lambda_{nk}(t) = \sum_{k=0}^{n+1} \tilde{\lambda}_{n+1,k}(t) - \tilde{\lambda}_{n+1,0}(t) \to 1 \quad \text{when } n \to \infty$$

because $\tilde{\lambda}_{n+1,0}(t) \to 0$. □

The following theorem is due to Borwein [4]. Borwein actually stated Theorem(1.13) with (1.24) replaced by

$$\lambda_{n+1} \leqslant c + \lambda_n \quad \text{for } n \geqslant n_0 \tag{1.23}$$

but his proof used only the weaker condition (1.24). Later we will see in Example 1.1 that it is possible to have (1.24) hold for some c but (1.23) fail for any c.

Theorem 1.13 *If $p \geqslant 1$, $c > 0$ and*

$$\mu = \sup_{0 \leqslant k \leqslant n} \frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k + c) \cdots (\lambda_{n-1} + c)} < \infty, \tag{1.24}$$

and if $\int_0^1 t^{-c/p}\, d\alpha(t) < \infty$, then $H(\lambda, \alpha) \in B(l_p)$, and

$$\|H(\lambda, \alpha)\|_p \leqslant \mu^{1/p} \int_0^1 t^{-c/p}\, d\alpha(t).$$

The results and examples in the remainder of this section can be found in D. Borwein and X. Gao [9]. First we introduce a lemma, which will be needed to prove the main theorem of this section.

Lemma 1.14 *Let $c > 0$. If $\liminf_{n \to \infty} \lambda_n / n > 0$, and $c > \limsup \lambda_n / n$, then*

$$\mu = \sup_{0 \leqslant k \leqslant n} \frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k + c) \cdots (\lambda_{n-1} + c)} < \infty.$$

PROOF: Let n_0 be a positive integer such that $\lambda_n / n \leqslant c$ when $n \geqslant n_0$. For $n \geqslant k > n_0$ we have

$$\frac{\lambda_k + c}{\lambda_k} \cdot \frac{\lambda_{k+1} + c}{\lambda_{k+1}} \cdots \frac{\lambda_{n-1} + c}{\lambda_{n-1}} \geqslant \left(1 + \frac{1}{k}\right) \cdots \left(1 + \frac{1}{n-1}\right) = \frac{n}{k},$$

so that

$$\frac{\lambda_{k+1}}{\lambda_k + c} \cdots \frac{\lambda_n}{\lambda_{n-1} + c} \leqslant \frac{k}{\lambda_k} \frac{\lambda_n}{n} \leqslant \left(\sup_{k \geqslant n_0} \frac{k}{\lambda_k}\right) c = M_1 < \infty$$

Let

$$M_2 = \max_{0 \leqslant k \leqslant m \leqslant n_0} \frac{\lambda_{k+1}}{\lambda_k + c} \cdots \frac{\lambda_m}{\lambda_{m-1} + c}.$$

Then for $0 \leqslant k \leqslant n_0 < n$ we have

$$\frac{\lambda_{k+1} \cdots \lambda_n}{(\lambda_k + c) \cdots (\lambda_{n-1} + c)} = \frac{\lambda_{k+1} \cdots \lambda_{n_0}}{(\lambda_k + c) \cdots (\lambda_{n_0 - 1} + c)} \cdot \frac{\lambda_{n_0 + 1} \cdots \lambda_n}{(\lambda_{n_0} + c) \cdots (\lambda_{n-1} + c)}$$

$$\leqslant M_2 M_1 < \infty,$$

and so $\mu \leqslant \max(M_1, M_2, M_2 M_1) < \infty$. $\square$

Theorem 1.15 *Suppose $p \geqslant 1$. Let $c_1 = \liminf \lambda_n/n$ and $c_2 = \limsup \lambda_n/n$.*

(i). *If $0 < c_1 < \infty$, $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$, and α is a non-decreasing function on $[0,1]$ su h that $\alpha(0+) = \alpha(0)$, then*

$$\|H(\lambda, \alpha)\|_p \geqslant \int_0^1 t^{-c_1/p}\, d\alpha(t).$$

In particular, if $\int_0^1 t^{-c_1/p}\, d\alpha(t) = \infty$, then $H(\lambda,\alpha) \notin B(l_p)$.

(ii). *If $\lim \lambda_n/n = \infty$ (i.e., $c_1 = \infty$), $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$, and α is a non-decreasing function on $[0,1]$ such that $\alpha(0+) = \alpha(0) < \alpha(r)$ for some $r \in (0,1)$, then $H(\lambda, \alpha) \notin B(l_p)$.*

(iii). *If $\int_0^1 t^{-c/p}\, |d\alpha(t)| < \infty$ for some $c > c_2$, and if $c_1 > 0$, then $H(\lambda,\alpha) \in B(l_p)$, and*

$$\|H(\lambda, \alpha)\|_p \leqslant \mu^{1/p} \int_0^1 t^{-c/p}\, |d\alpha(t)| < \infty$$

where μ is given by (1.24).

(iv). *If $\lim \lambda_n/n = 0$ and $\lim(\lambda_{n+1} - \lambda_n)$ exists, and if $\int_0^1 t^{-\epsilon}\, |d\alpha(t)| < \infty$ for some $\epsilon > 0$, then $H(\lambda,\alpha) \in B(l_p)$ for all $p \geqslant 1$.*

(v). *If the sequence $\{d_n\}$ given by (1.10) is eventually non-decreasing, and if for some fixed $p \geqslant 1$, $\int_0^1 t^{-1/p}\, |d\alpha(t)| < \infty$, then $H(\lambda,\alpha) \in B(l_p)$.*

PROOF:

(i). Let $0 < w < c_1/p$, let

$$b_n = \left(\frac{\lambda_1 \cdots \lambda_n}{(\lambda_1 + w) \cdots (\lambda_n + w)} \right)^p, \qquad \tilde{\lambda}_n = \lambda_n + w,$$

and define $\tilde{\lambda}_{nk}(t)$ by Definition(1.5) with $\{\tilde{\lambda}_n\}$ in place of $\{\lambda_n\}$. Since $wp < c_1$ there is a positive integer n_0 such that $\lambda_n \geqslant nwp$ foı all $n \geqslant n_0$. Hence, for $n \geqslant n_0$,

$$\frac{b_n}{b_{n-1}} = \frac{\lambda_n^p}{(\lambda_n + w)^p} = \left(1 + \frac{w}{\lambda_n}\right)^{-p} \geqslant 1 - \frac{pw}{\lambda_n} \geqslant 1 - \frac{1}{n} = \frac{n-1}{n},$$

and so $b_n \geqslant b_{n_0} n_0/n$. It follows that $\sum b_n = \infty$. Further, for $0 \leqslant k \leqslant n, 0 < t \leqslant 1$,

$$\lambda_{nk}(t)\left(\frac{b_k}{b_n}\right)^{1/p} = -(\lambda_{k+1} + w)\cdots(\lambda_n + w)\frac{1}{2\pi i}\int_C \frac{t^z\,dz}{(\lambda_k - z)\cdots(\lambda_n - z)}$$

$$= -\tilde{\lambda}_{k+1}\cdots\tilde{\lambda}_n\frac{t^{-w}}{2\pi i}\int_{C_1}\frac{t^{z_1}\,dz_1}{(\tilde{\lambda}_k - z_1)\cdots(\tilde{\lambda}_n - z_1)} \qquad (z_1 = z + w)$$

$$= \tilde{\lambda}_{nk}(t)t^{-w}.$$

By Lemma(1.12) and Fatou's theorem,

$$\liminf_{n\to\infty}\sum_{k=0}^{n}\lambda_{nk}\left(\frac{b_k}{b_n}\right)^{1/p} = \liminf_{n\to\infty}\sum_{k=0}^{n}\int_0^1 \tilde{\lambda}_{nk}(t)t^{-w}\,d\alpha(t) \geqslant \int_0^1 t^{-w}\,d\alpha(t),$$

and hence, by Lemma(1.6),

$$\|H(\lambda,\alpha)\|_p \geqslant \int_0^1 t^{-w}\,d\alpha(t). \tag{1.25}$$

It follows, on letting $w \to c_1/p$ from the left and appealing to the monotone convergence theorem, that

$$\|H(\lambda,\alpha)\|_p \geqslant \int_0^1 t^{-c_1/p}\,d\alpha(t). \tag{1.26}$$

Notice that (1.26) is true when the right side is either finite or infinite. In the case when it is infinite, (1.26) means exactly that $H(\lambda,\alpha) \notin B(l_p)$.

(ii). For any $w > 0$, we obtain (1.25) exactly as in part (i). It follows that $\|H(\lambda,\alpha)\|_p \geqslant \int_0^r t^{-w}\,d\alpha(t) \geqslant r^{-w}\int_0^r d\alpha(t) \to \infty$ as $w \to \infty$. Thus $H(\lambda,\alpha) \notin B(l_p)$.

(iii). This is an immediate consequence of Lemma(1.14) and Theorem(1.13).

(iv). Observe that

$$\frac{\lambda_n}{n} = \frac{1}{n}\sum_{k=0}^{n-1}(\lambda_{k+1} - \lambda_k) + \frac{\lambda_0}{n}.$$

Because $\lim \lambda_n/n = 0$ and $\lim(\lambda_{n+1} - \lambda_n)$ exists, we see by a routine argument that $\lim(\lambda_{n+1} - \lambda_n) = 0$. Hence $\lambda_{n+1} \leqslant \lambda_n + \epsilon p$ for large n, and so, by Theorem(1.13), $H(\lambda, \alpha) \in B(l_p)$.

(v). This also follows from Theorem(1.13) since, by (1.10),

$$\lambda_{n+1} - \lambda_n = \frac{D_{n+1}}{d_{n+1}} - \frac{D_n}{d_n} = D_n\left(\frac{1}{d_{n+1}} - \frac{1}{d_n}\right) + 1 \leqslant 1$$

for large n. □

Recall that for a sequence $a = \{a_n\}$ with $a_0 > 0$ and $a_n = 0$, M_a denotes the weighted mean matrix as defined in Definition(1.3).

Corollary 1.16 *Suppose $p \geqslant 1$. Let $c_1 = \liminf_{n\to\infty} A_n/na_n$ and $c_2 = \limsup_{n\to\infty} A_n/na_n$.*

(i). *If either $\sum a_n$ is convergent, or $p \leqslant c_1 \leqslant \infty$, then $M_a \notin B(l_p)$.*

(ii). *If $0 < c_1 \leqslant c_2 < p$, then $M_a \in B(l_p)$ and*

$$\frac{p}{p-c_1} \leqslant \|M_a\|_p \leqslant \mu^{1/p}\frac{p}{p-c_2} < \infty,$$

where μ is given by (1.24) with $\lambda_n = \dfrac{A_n}{a_n} - 1$ and any $c \in (c_2, p)$. Furthermore, if $0 < \lim A_n/na_n = c < p$ and

$$\frac{A_{n+1}}{a_{n+1}} \leqslant c + \frac{A_n}{a_n} \text{ for } n \geqslant 0, \tag{1.27}$$

then $\|M_a\|_p = p/(p-c)$.

(iii). *If $\lim A_n/na_n = 0$, and*

$$\frac{A_{n+1}}{a_{n+1}} - \frac{A_n}{a_n}$$

tends to a limit, then $M_a \in B(l_p)$ *for all* $p \geqslant 1$.

(iv). *If* $\lim A_n/na_n = 0$, *and* $\{a_n\}$ *is eventually monotonic, then* $M_a \in B(l_p)$ *for all* $p > 1$.

Remark. This corollary generalized the following theorem of Cass and Kratz [21, Theorem 2] by removing the logarithmico-exponential condition.

Theorem 1.17 (Cass and Kratz) *Suppose that* $p > 1$ *and* $a_n = f(n)$ *where* $f(x)$ *is a logarithmico-exponential function for* $x > x_0$, *and that* $A_n/na_n \to c$ *(this limit always exists when* $a_n = f(n)$, *as proved in [21, Lemma 3]). Then* $M_a \in B(l_p)$ *iff* $c < p$, *in which case*

$$\frac{p}{p-c} \leqslant \|M_a\|_p \leqslant \sigma_1^{1/p'} \sigma_2^{1/p} < \infty$$

where

$$\sigma_1 = \sup_{n \geqslant 0} \sum_{k=0}^{n} \frac{a_k}{A_n} \left(\frac{n+1}{k+1} \right)^{1/p} \quad \textit{and} \quad \sigma_2 = \sup_{k \geqslant 0} \sum_{n=k}^{\infty} \frac{a_k}{A_n} \left(\frac{k+1}{n+1} \right)^{1/p'} .$$

PROOF OF COROLLARY(1.16):

(i). If $\sum a_n < \infty$, then $\sum (a_0/A_n)^p = \infty$; but this implies that $M_a e_0 \notin l_p$, so that $M_a \notin B(l_p)$. That $M_a \notin B(l_p)$ when $\infty \geqslant c_1 \geqslant p$ and $\sum a_n = \infty$ follows directly from parts (i) and (ii) of Theorem(1.15), with $\alpha(t) = t$ and $\lambda_n = A_n/a_n - 1$, since $A_n \to \infty$ if and only if $\sum_{n=1}^{\infty} 1/\lambda_n = \infty$

(ii). This follows from parts (i) and (iii) of Theorem(1.15), the final conclusion being justified because the appropriate $\mu = 1$ when (1.27) holds.

(iii). This follows from Theorem(1.15)(iv) and Lemma(1.4), since

$$\frac{A_{n+1}}{a_{n+1}} - \frac{A_n}{a_n} = \lambda_{n+1} - \lambda_n,$$

and $\int_0^1 t^{-\epsilon}\,dt < \infty$ when $\epsilon < 1$.

(iv). First we prove that the hypothesis $\lim A_n/na_n = 0$ implies that $\lim a_n = \infty$. (In fact the hypothesis implies that $n^{-c}a_n \to \infty$ for every real constant c). Let $\alpha_n = na_n/A_n$. Then

$$\log \frac{A_n}{A_{n-1}} = -\log\left(1 - \frac{\alpha_n}{n}\right) > \frac{\alpha_n}{n},$$

so that $\log A_n > \log A_0 + \sum_{k=1}^{n} \alpha_k/k$. Since $\alpha_n \to \infty$, it follows that $\log A_n/\log n \to \infty$, and hence that, for any given c and sufficiently large n, $\log A_n > (c+1)\log n$, or $A_n > n^{c+1}$. Therefore $n^{-c}a_n = n^{-c-1}\alpha_n A_n > \alpha_n \to \infty$ as $n \to \infty$.

Hence, since $\{a_n\}$ is eventually monotonic it must be eventually non-decreasing, and so $M_a \in B(l_p)$ for every $p > 1$ by Theorem(1.15)(v). □

We close this section with two examples. For the first we construct a generalized Hausdorff matrix $H(\lambda, \alpha) \in B(l_{p_0})$, with α a positive measure on $[0,1]$, for which $\int_0^1 t^{-c/p_0}\,d\alpha(t) = \infty$ for every $c > c_2$, where $p_0 \geqslant 1$ and c_2 is as in Theorem(1.15)(iii). This will show that the conclusion of Theorem(1.15)(iii) can hold when its main condition is not satisfied. The example will also show that (1.24) can hold with $c = 1$ while (1.23) fails for any c.

Example 1.1

Suppose that $p_0 > 1$. Let

$$\lambda_n = m^2 \quad \text{for } m^2 \leqslant n < (m+1)^2, m = 0, 1, 2, \ldots,$$

$$d\alpha(t) = \frac{t^{1/p_0}}{t \log^2 \frac{t}{2}}\,dt \quad \text{for } t \in (0, 1].$$

Observe that $c_2 = \lim \lambda_n/n = 1$, and $\int_0^1 t^{-c/p_0}\,d\alpha(t) = \infty$ for all $c > c_2$. Thus Theorem(1.15)(iii) cannot be used to prove that $H(\lambda, \alpha) \in B(l_{p_0})$. Instead we shall appeal to Theorem(1.13). For $m \geqslant 1$, we have that

$$\beta_m = \frac{\lambda_{m^2+1}\cdots\lambda_{(m+1)^2}}{(\lambda_{m^2}+1)\cdots(\lambda_{m^2+2m}+1)} = \left(\frac{m^2}{m^2+1}\right)^{2m+1}\left(\frac{m+1}{m}\right)^2.$$

Since

$$\left(\frac{m^2+1}{m^2}\right)^{m+\frac{1}{2}} = \left(1+\frac{1}{m^2}\right)^{m+\frac{1}{2}} \geqslant 1+\left(m+\frac{1}{2}\right)\frac{1}{m^2} > 1+\frac{1}{m} = \frac{n.+1}{m},$$

it follows that $\beta_m \leqslant 1$ for $m \geqslant 1$. Also because

$$\lim_{m\to\infty}\frac{\lambda_{m^2}}{\lambda_{m^2-1}+1} = 1,$$

and $\lambda_k/(\lambda_{k-1}+1) < 1$ when k is not a perfect square, we get

$$\sup_{0\leqslant k\leqslant n}\frac{\lambda_{k+1}\cdots\lambda_n}{(\lambda_k+1)\cdots(\lambda_{n-1}+1)} < \infty.$$

Since

$$\int_0^1 t^{-1/p_0}\,d\alpha(t) = \int_0^1 \frac{dt}{t\log^2\frac{t}{2}} < \infty,$$

it follows that $H(\lambda,\alpha) \in B(l_{p_0})$ by Theorem(1.13) with $c = 1$. In fact this theorem shows that $H(\lambda,\alpha) \in B(l_p)$ for all $p \geqslant p_0$. On the other hand, a simple consequence of Theorem(1.15)(i) is that $H(\lambda,\alpha) \notin B(l_p)$ for $1 < p < p_0$.

Finally, we see that (1.24) holds with $c = 1$, but that (1.23) cannot hold for any c since $\limsup_{n\to\infty}(\lambda_{n+1}-\lambda_n) = \infty$.

The second example will show us that it is possible for $M_a \notin B(l_p)$ when $A_n/na_n \to 0$, although Corollary(1.16) tells us that $M_a \in B(l_p)$ when $A_n/na_n \to c \in (0,p)$. Thus the condition $A_n/na_n \to 0$ needs to be augmented, as in parts (iii) and (iv) of Corollary(1.16), in order to ensure that $M_a \in B(l_p)$. Correspondingly, the condition $\lambda_n/n \to 0$ needs to be augmented, as in parts (iv) of Corollary(1.16), in order to ensure that $H(\lambda,\alpha) \in B(l_p)$.

Example 1.2

Define the weighted mean matrix M_a with $a = \{a_n\}$ as follows:

$$a_n = \begin{cases} 1 & \text{for } n = 0 \\ 2^m & \text{for } m^2 < n < (m+1)^2, m = 1, 2, \ldots \\ m2^{m+1} & \text{for } n = m^2. \end{cases}$$

Then

$$a_{m^2} = \sum_{k=m^2+1}^{(m+1)^2-1} a_k,$$

and so the partial sum

$$A_{(m+1)^2-1} = 1 + \sum_{k=1}^{m} k2^{k+2} = (m-1)2^{m+3} + 9.$$

Hence, for $m^2 \leqslant n < (m+1)^2$,

$$\frac{na_n}{A_n} \geqslant \frac{m^2 2^m}{(m-1)2^{m+3}+9} \to \infty, \text{ and } \frac{a_{m^2}}{A_n} \geqslant \delta_m = \frac{m2^{m+1}}{(m-1)2^{m+3}+9} \to \frac{1}{4}.$$

Now let $p > 1$, and define $x = \{x_k\} \in l_p$ by setting

$$x_k = \begin{cases} \dfrac{1}{m^{1/p}\log m} & \text{if } k = m^2, m = 2, 3, \ldots \\ 0 & \text{otherwise,} \end{cases}$$

and let

$$y_n = \frac{1}{A_n}\sum_{k=0}^{n} a_k x_k.$$

Then, for $4 \leqslant m^2 \leqslant n < (m+1)^2$,

$$y_n \geqslant \frac{a_{m^2}}{A_n} x_{m^2} \geqslant \frac{\delta_m}{m^{1/p}\log m}.$$

Thus

$$\sum_{n=m^2}^{(m+1)^2-1} y_n^p \geqslant \frac{2m\delta_m^p}{m\log^p m}, \text{ and so } \sum_{n=0}^{\infty} y_n^p \geqslant \sum_{m=2}^{\infty} \frac{2\delta_m^p}{\log^p m} = \infty.$$

Consequently $M_a \notin B(l_p)$, even though $\lim A_n/na_n = 0$.

Chapter 2

Non-negative Matrices in (l_p, l_q)

The results in this chapter will appear in [10].

2.1 Introduction

A matrix $A = (a_{nk})_{n,k=1}^{\infty}$ is said to be nonnegative if $a_{nk} \geqslant 0$ for $n, k \in \mathbf{N}$. To avoid trivial cases we assume in all that follows that no matrix A is identically zero. Similarly, a sequence $u = (u_j)$ is said to be non-negative (resp. positive) if $u_j \geqslant 0$ (resp. $u_j > 0$) for all $j \in \mathbf{N}$. In such cases we write $u \geqslant 0$ (resp. $u > 0$). Like the definition of p', we define $q' = q/(q-1)$ for $q > 1$.

First of all, the cases of $p < q$ and $p > q$ are very different. By a theorem of H. R. Pitt, all bounded linear operators from l_p to l_q are compact when $p > q$. A proof of this can be found in [33, p.76] and it can also be proved directly using a Schur sliding hump technique.

In the special case where the entries in a trangular matrix $A = (a_{nk})$ decompose so that

$$a_{nk} = \begin{cases} a_n b_k, & k \leqslant n, \\ 0 & k > n, \end{cases} \tag{2.1}$$

where $\{a_n\}_{n=1}^{\infty}$ and $\{b_k\}_{k=1}^{\infty}$ are two sequences of non-negative numbers, there are the following characterizations due to Bennett [1]. Similar results where sequences are replaced by functions or measures can be found in Muckenhoupt [37] and others.

Theorem 2.1 (G. Bennett) *For a matrix A given by (2.1), $A \in (l_p, l_q)$ if and only if*

(i). $\left(\sum_{n=N}^{\infty} a_n^q\right)^{1/q} \left(\sum_{k=1}^{N} b_k^{p'}\right)^{1/p'} = O(1)$ *for* $1 < p \leqslant q < \infty$*;*

(ii). $\sum_{n=1}^{\infty} a_n^q \left(\sum_{m=n}^{\infty} a_m^q\right)^{r/p} \left(\sum_{k=1}^{n} b_k^{p'}\right)^{r/p'} < \infty$ *for* $1 < p < \infty$ *and* $q < p$.

We will discuss tractable necessary and sufficient conditions for a non-negative matrix A to be in (l_p, l_q) when $1 < q \leqslant p$. We will also give alternative proofs for some known results, thereby filling a gap in the proof of Theorem(2.5) in §2.2, which is the case $p = q$ of Koskela's Theorem 1 in [31]. The first part of Theorem(2.5) was originally proved by Ladyženskiĭ in [32]. In the case where $1 < q < p < \infty$ Koskela's result will be refined, and a weakened form of the Vere-Jones conjecture in [46] concerning matrix operators on l_p will also be discussed.

2.2 Alternative Proofs and Refinements of Some Known Results

Different cases of the following theorem will be refined by Theorems(2.5), (2.6) and (2.8). As it is a necessary step in establishing these later results, we will prove it ahead of others.

Theorem 2.2 *Suppose that $p, q > 1$, that the non-negative matrix $A = (a_{ij}) \in (l_p, l_q)$, and that $C > (\|A\|_{p,q})^q$. Then there exists a positive sequence $\mathbf{u} = (u_j)$ such that $\|u\|_p \leqslant 1$ and*

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{q-1} \leqslant C u_j^{p-1}, \qquad j = 1, 2, \cdots. \tag{2.2}$$

I am indebted to Dr. G. Sinnamon for adapting Lemmas(2.3), (2.4), and the proof of Theorem(2.2), from Szeptycki's work in [43, Theorem 7.1.6].

In order to prove Theorem(2.2) we first adopt some notation. We define $B_p^+ = \{u \mid u \in l_p, u \geqslant 0, \text{ and } \|u\|_p \leqslant 1\}$, and define E_r a mapping from the set of non-negative sequences to itself by $E_r u = (u_j^r)$.

Lemma 2.3 *Let $p \geqslant 1$. Suppose that S is a continuous, order preserving map from B_p^+ to B_p^+ and that $0 < t < 1$. Then there exists a positive $u \in B_p^+$ such that $tSu < u$.*

PROOF: Choose $u^{(1)} \in B_p^+$ such that $u^{(1)} > 0$ and $\|u^{(1)}\|_p = 1 - t$. For $n \in \mathbf{N}$, define $u^{(n+1)} = u^{(1)} + tSu^{(n)} > 0$. Note that if $u^{(n)} \in B_p^+$ for any $n \in \mathbf{N}$, then $Su^{(n)} \in B_p^+$, so $\|u^{(n+1)}\|_p \leqslant \|u^{(1)}\|_p + t\|Su^{(n)}\|_p \leqslant 1 - t + t = 1$, and hence $u^{(n+1)} \in B_p^+$. Also $u^{(2)} - u^{(1)} = tSu^{(1)} \geqslant 0$, and if $u^{(n+1)} - u^{(n)} \geqslant 0$ for any $n \in \mathbf{N}$, then $u^{(n+2)} - u^{(n+1)} = t(Su^{(n+1)} - Su^{(n)}) \geqslant 0$ since S is order preserving. It follows that the sequence of sequences $(u^{(n)})$ is term-wise non-decreasing in B_p^+. By the monotone convergence theorem, u, the term-wise limit of $u^{(n)}$, is also the l_p-limit of $(u^{(n)})$. Hence $u \in B_p^+$, and since S is continuous, we have that $u = u^{(1)} + tSu > tSu \geqslant 0$ as required. □

Lemma 2.4 *Let $p \geqslant 1$ and $r > 0$. Then E_r is a continuous , order preserving map from B_{pr}^+ to B_p^+.*

PROOF: Only the continuity of E_r is not immediately evident.

Let $x = (x_j)$ and $y = (y_j)$ be sequences in B_{pr}^+.

If $r \leqslant 1$, then $|x_j^r - y_j^r| \leqslant |x_j - y_j|^r$ by basic calculus, and so

$$\|E_r x - E_r y\|_p = (\sum_{j=1}^{\infty} |x_j^r - y_j^r|^p)^{1/p} \leqslant (\sum_{j=1}^{\infty} |x_j - y_j|^{pr})^{1/p} = (\|x - y\|_{pr})^r.$$

If $r > 1$, then $|x_j^r - y_j^r| \leqslant r(x_j + y_j)^{r-1}|x_j - y_j|$ by the mean value theorem, and so, using Hölder's and Minkowski's inequalities, we get the estimate

$$\|E_rx - E_ry\|_p = (\sum_{j=1}^{\infty} |x_j^r - y_j^r|^p)^{1/p} \leqslant (\sum_{j=1}^{\infty} r^p(x_j + y_j)^{p(r-1)}|x_j - y_j|^p)^{1/p}$$

$$\leqslant r(\sum_{j=1}^{\infty}(x_j + y_j)^{pr})^{(r-1)/pr}(\sum_{j=1}^{\infty} |x_j - y_j|^{pr})^{1/pr} \leqslant r(\|x\|_{pr} + \|y\|_{pr})^{r-1}\|x - y\|_{pr}.$$

It follows that E_r is a continuous map from B_{pr}^+ to B_p^+. □

PROOF OF THEOREM(2.2): Since $\|A\|_{p,q} > 0$, we can divide A by $\|A\|_{p,q}$ and thereby reduce the problem to the case $\|A\|_{p,q} = 1$. Note that the transpose matrix A^* satisfies $\|A^*\|_{q',p'} = 1$. Further, A is a continuous, order preserving map from B_p^+ to B_q^+ and A^* is an order preserving map from $B_{q'}^+$ to $B_{p'}^+$. Therefore

$$S = E_{p'/p}A^*E_{q/q'}A$$

is a continuous, order preserving map from B_p^+ to B_p^+. Let $0 < t < 1$. Then, by Lemma(2.3), there is a positive $u = (u_j) \in B_p^+$ such that $tSu < u$, that is

$$t\left(\sum_{i=1}^{\infty} a_{ij}(\sum_{k=1}^{\infty} a_{ik}u_k)^{q/q'}\right)^{p'/p} < u_j, \qquad j = 1, 2, \cdots,$$

and therefore

$$\sum_{i=1}^{\infty} a_{ij}(\sum_{k=1}^{\infty} a_{ik}u_k)^{q-1} < t^{1-p}u_j^{p-1}, \qquad j = 1, 2, \cdots. \qquad \square$$

The combination of Theorems(2.5) and (2.6) is Theorem 1 of Koskela[31]. Koskela stated his Theorem 1 with the sequence u non-negative throughout and with the following additional property:

$$u_j = 0 \textit{ if and only if } a_{ij} = 0 \textit{ for every } i \in \mathbf{N}. \tag{2.3}$$

Note that u_j can be given any positive value when the j-th column of A is identically zero without affecting the validity of (2.2) So our theorems are equivalent to Theorem 1 of [31].

Theorem 2.5 *Let $p > 1$. Then a non-negative matrix $A = (a_{ij}) \in (l_p, l_p)$ if and only if there exist a positive number C and a positive sequence $u = (u_j)$ such that*

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{p-1} \leqslant C u_j^{p-1}, \qquad j = 1, 2, \cdots, \tag{2.4}$$

and then $\|A\|_p \leqslant C^{1/p}$. Further, if the non-negative matrix $A \in (l_p, l_p)$, then there exists a positive sequence u for which (2.4) holds with $C = (\|A\|_p)^p$.

Theorem 2.6 *Let $p > q > 1$. Then a non-negative matrix $A = (a_{ij}) \in (l_p, l_q)$ if and only if there exist a positive constant C and a positive sequence $u = (u_j)$ such that $\|u\|_p \leqslant 1$ and (2.2) holds, and then $C^{1/q} \geqslant \|A\|_{p,q}$. Further, if the non-negative matrix $A \in (l_p, l_q)$, then there exists a non-negative sequence $u = (u_j)$ with $0 < \|u\|_p \leqslant 1$ for which (2.2) holds with $C = (\|A\|_{p,q})^q$, and $u_j = 0$ only when $a_{ij} = 0$ for every $i \in \mathbf{N}$.*

Remark. As pointed out in [10], the original proof in [31] of the necessity part of Theorem(2.5) is flawed. When A is positive, then a positive sequence (u_j) satisfying (2.4) can be found by the method used in [31]. For a general non-negative matrix A, Koskela suggests first applying the result to $A + \epsilon B$, where $\epsilon > 0$ and B is a fixed positive matrix in (l_p, l_p), to obtain a positive sequence $(u_j^{(\epsilon)})$ satisfying the appropriate version of (2.4), and then using a simple continuity argument. He gives no indication, however, of how to prevent $\liminf_{\epsilon \to 0+} u_j^{(\epsilon)}$ from being infinite or 0. There is a similar flaw in the proof of the necessity part of Theorem(2.6) in [31], although it is very easy to fix.

Here we provide alternative proofs of the necessity parts of theorems (2.5) and (2.6), which show, *inter alia*, how to avoid the sort of difficulty mentioned above. We introduce an equivalence relation "$\sim$", and prove an additional lemma.

Given a non-negative matrix $A = (a_{ij})$, let $\mathbf{N}_+$ be the set of positive integers j such that $a_{ij} > 0$ for some $i \in \mathbf{N}$. We define an equivalence relation "$\sim$" on $\mathbf{N}_+$ as follows:

For $j, k \in \mathbf{N}_+$, we say that $j \sim k$ if either $j = k$, or there is a chain of distinct positive integers $j_1, j_2, \ldots, j_{r-1}, j_r$ such that $j = j_1$, $k = j_r$ and, for each $\nu \in \{1, 2, \cdots, r-1\}$, there is an $i_\nu \in \mathbf{N}$ such that $a_{i_\nu j_\nu} > 0$ and $a_{i_\nu j_{\nu+1}} > 0$.

Lemma 2.7 *Let $p > 1$. Suppose that $A = (a_{ij})$ is a non-negative matrix, that $C > 0$, and that $u = (u_j)$ is a positive sequence satisfying (2.4) Then*

(i). *for any fixed $m \in \mathbf{N}_+$ and $a > 0$, (2.4) continues to hold if u is replaced by $v = (v_j)$ where*

$$v_j = \begin{cases} u_j & \text{if } j \not\sim m, \\ au_j & \text{if } j \sim m; \end{cases}$$

(ii). *for fixed $j, k \in \mathbf{N}_+$ with $j \sim k$ and $j \neq k$, there is a positive integer r and positive constants K_1, K_2 such that*

$$K_1 C^{-r(p'-1)} u_k \leqslant u_j \leqslant K_2 C^{r(p'-1)} u_k.$$

PROOF:

(i). Let $\mathbf{N}_m = \{k \in \mathbf{N}_+ \mid k \sim m\}$. Then

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} v_k \Big)^{p-1} = \sum_{i=1}^{\infty} \Big(\sum_{k \in \mathbf{N}_m} a_{ij}^{p'-1} a_{ik} a u_k + \sum_{k \notin \mathbf{N}_m} a_{ij}^{p'-1} a_{ik} u_k \Big)^{p-1},$$

from which the desired result follows since

$$\sum_{k \notin \mathbf{N}_m} a_{ij}^{p'-1} a_{ik} u_k = 0 \text{ when } j \sim m, \text{ and } \sum_{k \in \mathbf{N}_m} a_{ij}^{p'-1} a_{ik} a u_k = 0 \text{ when } j \not\sim m.$$

(ii). Let $j = j_1, j_2, \ldots, j_{r-1}, j_r = k$ be the chain of integers and $i_1, i_2, \cdots, i_{r-1}, i_r$ the corresponding indices of the definition of $j \sim k$. It follows from (2.4) that, for $\nu = 1, 2, \cdots, r-1$,

$$u_{j_\nu}^{p-1} \geqslant C^{-1} \sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{p-1} \geqslant C^{-1} a_{i_\nu j_\nu} a_{i_\nu j_{\nu+1}}^{p-1} u_{j_{\nu+1}}^{p-1} > 0.$$

Combining these inequalities we see that $u_j \geqslant K_1 C^{-r(p'-1)} u_k$ for some positive constant K_1. Likewise there is a positive constant K_2 such that $u_k \geqslant K_2^{-1} C^{-r(p'-1)} u_j$. □

PROOF OF THEOREM(2.5) AND THEOREM(2.6):

Sufficiency: Assume that C is a positive number and that $u_j > 0$ for all $j \in \mathbf{N}$, $\|u\|_p \leqslant 1$ if $p > q$, and depending on the theorem we want to prove, (2.4) or (2.2) holds. Let $x = (x_k)_{k=1}^{\infty} \in l_p$, by Hölder's inequality,

$$\left|\sum_{j=1}^{\infty} a_{ij} x_j\right|^q = \left|\sum_{j=1}^{\infty} a_{ij}^{1/q} u_j^{-1/q'} x_j\, a_{ij}^{1/q'} u_j^{1/q'}\right|^q \leqslant \left(\sum_{j=1}^{\infty} a_{ij} u_j^{1-q} |x_j|^q\right)\left(\sum_{k=1}^{\infty} a_{ik} u_k\right)^{q-1},$$

$i = 1, 2, \ldots$. Combining this with (2.2) we get

$$\|A(x)\|_q^q \leqslant C \sum_{j=1}^{\infty} u_j^{p-q} |x_j|^q.$$

If $p = q$ this already gave us $\|A\|_{p,q} \leqslant C^{1/q}$. If $p > q$, remembering $\|u\|_p \leqslant 1$, another application of Hölder's inequality with the pair $p_1 = p/q, p_1' = p/(p-q)$ yields

$$\sum_{j=1}^{\infty} u_j^{p-q} |x_j|^q \leqslant \|u\|_p^{p-q} \|x\|_p^q \leqslant \|x\|_p^q,$$

thus we also have $\|A\|_{p,q} \leqslant C^{1/q}$.

Necessity: Suppose that $p \geqslant q > 1$ and that the non-negative matrix $A = (a_{ij}) \in (l_p, l_q)$. Let $C_n = (\|A\|_{p,q})^q + 1/n$ for $n \in \mathbf{N}$. Then, by Theorem(2.2), there is a positive sequence $u^{(n)} = (u_j^{(n)})$ such that $\|u^{(n)}\|_p \leqslant 1$ and

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k^{(n)}\Big)^{q-1} \leqslant C_n (u_j^{(n)})^{p-1}, \qquad j = 1, 2, \cdots. \tag{2.5}$$

Case 1. Let $p > q > 1$. Define $u = (u_j)$ where $u_j = \liminf_{n\to\infty} u_j^{(n)}$. Then $\|u\|_p \leqslant 1$ and

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k\Big)^{q-1} \leqslant (\|A\|_{p,q})^q u_j^{p-1}, \qquad j = 1, 2, \cdots. \tag{2.6}$$

Hence, for every $i \in \mathbf{N}$, $a_{ij}^q \leqslant (\|A\|_{p,q})^q u_j^{p-q}$. It follows that $u_j > 0$ whenever $j \in \mathbf{N}_+$. The above process could yield $u_j = 0$, but only when $j \in \mathbf{N} \setminus \mathbf{N}_+$, that is, when the j-th column of A is identically 0. This establishes Theorem(2.6).

Case 2. Let $p = q > 1$. Let $\mathbf{N}'$ be the set of first elements in the equivalence classes associated with the equivalence relation "$\sim$" on $\mathbf{N}_+$. For each $k \in \mathbf{N}'$ and $j \sim k$ divide $u_j^{(n)}$ by $u_k^{(n)}$ which, by Lemma(2.7)(i), we can do without affecting the validity of (2.5). Thus we now have $u_k^{(n)} = 1$ for all $k \in \mathbf{N}'$. Also, by Lemma(2.7)(ii), we have, for fixed distinct $j, k \in \mathbf{N}_+$ with $k \in \mathbf{N}'$ and $j \sim k$, that there is a positive integer r and positive constants K_1, K_2 such that

$$K_1 C_n^{-r(p'-1)} \leqslant u_j^{(n)} \leqslant K_2 C_n^{r(p'-1)}.$$

Define

$$u_j = \begin{cases} \liminf\limits_{n\to\infty} u_j^{(n)} & \text{for } j \in \mathbf{N}_+, \\ 1 & \text{for } j \in \mathbf{N} \setminus \mathbf{N}_+. \end{cases}$$

Then $\infty > u_j > 0$ for all $j \in \mathbf{N}$, and $u = (u_j)$ is a positive sequence satisfying (2.4) with $C = (\|A\|_p)^p$. Note that, for $j \in \mathbf{N} \setminus \mathbf{N}_+$, we could have defined u_j to be any positive number. This completes the proof of Theorem(2.5). □

Here is a refinement of Theorem(2.6).

Theorem 2.8 *Let $1 < q < p < \infty$, and suppose that the non-negative matrix $A = (a_{ij}) \in (l_p, l_q)$. Then there exists a non-negative sequence $u = (u_j)$ having the following properties:*

(a). $\|u\|_p = 1$;

(b). $u_j = 0$ *only if* $a_{ij} = 0$ *for every* $i \in \mathbf{N}$;

(c). *for each* $j \in \mathbf{N}$,

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{q-1} = \|A\|_{p,q}^q u_j^{p-1}. \tag{2.7}$$

PROOF: Let $C = (\|A\|_{p,q})^q$, and define $fu = (f_j u)$ where

$$f_j u = \sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{q-1} \text{ for } j \in \mathbf{N}.$$

By Theorem(2.6), there exists a non-negative sequence $u = (u_j)$ such that $0 < \|u\|_p \leqslant 1$ and $fu \leqslant Cu^{p-1}$, C being the smallest possible constant for which such an inequality can hold. If $\|u\|_p < 1$, then pick a constant $t > 1$ satisfying $t\|u\|_p \leqslant 1$ and let $v = tu$. We then get $fv \leqslant t^{q-1}Cu^{p-1} = t^{q-p}Cv^{p-1}$, which is a contradiction because $t^{q-p} < 1$. This proves that $\|u\|_p = 1$.

Now assume that there exists $j_0 \in \mathbf{N}$ such that $f_{j_0} u < Cu_{j_0}^{p-1}$, then on replacing u_{j_0} by λu_{j_0} for a λ less than and close enough to 1, we get a new u with p-norm less than 1 for which $fu \leqslant Cu^{p-1}$. But this is impossible by what we proved in the previous paragraph. Hence we must have $f_j u = Cu_j^{p-1}$ for every $j \in \mathbf{N}$. $\square$

That Theorem(2 8) does not hold for $p = q$ is shown by the following example involving a matrix A with no zero columns:

Let $p = q > 1$, and let $A = (a_{ij})$ with $a_{11} = 2$, $a_{ii} = 1$ for $i = 2, 3, \cdots$, and all other $a_{ij} = 0$. Then, for $j = 1, 2, \cdots$,

$$\sum_{i=1}^{\infty} a_{ij} \Big(\sum_{k=1}^{\infty} a_{ik} u_k \Big)^{p-1} = \lambda_j u_j^{p-1}$$

where $\lambda_1 = 2^p$ and $\lambda_j = 1$ for $j = 2, 3, \cdots$. So it is impossible to have a positive sequence $u = (u_j)$ satisfying (2.7).

2.3 Weakened Form of the Vere-Jones Conjecture

The sufficiency part of Theorem(1.5) in §1.1 has been very effective in establishing conditions for standard summability matrices to be in (l_p, l_p), as can be evidently seen from §1.2 and §1.3. The adjunct to it, Lemma(1.6), has also proved to be quite useful in showing $A \notin B(l_p)$.

In [46], Vere-Jones made the following:

Conjecture 2.1 (i). *A non-negative matrix $A = (a_{ij})_{i,j=1}^{\infty}$ is in $B(l_p)$ $(1 < p < \infty)$ if and only if there exist a sequence $b = (b_j)_{j=1}^{\infty}$ of positive numbers and a number M, $0 < M < \infty$ such that*

$$\left.\begin{array}{ll} \sum_{j=1}^{\infty} a_{ij}\,(b_j/b_i)^{1/p} \leqslant M, & i = 1,2,\ldots \\ \sum_{i=1}^{\infty} a_{ij}\,(b_i/b_j)^{1/p'} \leqslant M, & j = 1,2,\ldots \end{array}\right\}. \tag{2.8}$$

(ii). *Moreover, the norm of the operator defined by A can be identified with the least number M for which such a sequence b can be found.*

As a corollary to Theorem(2.5), Koskela in [31] proved the first part of this conjecture to be true. Thus the condition in Theorem(1.5) is in fact both necessary and sufficient:

Proof of (i) of Conjecture(2.1) (necessity part of Theorem(1.5)): Apply Theorem(2.5) to $A + E$, where E stands for the identity matrix, we have a sequence (u_j) of positive numbers and a $C > 0$ such that

$$\sum_{i=1}^{\infty} a_{ij}\Big(\sum_{k=1}^{\infty} a_{ik}u_k + u_i\Big)^{p-1} + \Big(\sum_{k=1}^{\infty} a_{jk}u_k + u_j\Big)^{p-1} \leqslant C u_j^{p-1}, \qquad j = 0,1,2,\cdots.$$

thus

$$\sum_{i=1}^{\infty} a_{ij}u_i^{p-1} \leqslant C u_j^{p-1} \quad \text{and} \quad \sum_{k=1}^{\infty} a_{jk}u_k \leqslant C^{p'-1}u_j.$$

Now, letting $b_k = u_k^p$, we get (2.8). □

Koskela also proved part (ii) of the Conjecture(2.1) false. But taking into account that two constants M_1 and M_2 were used in Theorem(1.5) to bound the norm of A, A more compelling version of the conjecture would seem to be:

Conjecture 2.2 *If $p > 1$ and the non-negative matrix $A \in (l_p, l_p)$, then there exists a positive sequence $b = \{b_j\}_{j=1}^{\infty}$ for which*

$$\left.\begin{array}{ll} \sum_{j=1}^{\infty} a_{ij}\,(b_j/b_i)^{1/p} \leqslant M_1, & i = 1,2,\ldots \\ \sum_{i=1}^{\infty} a_{ij}\,(b_i/b_j)^{1/p'} \leqslant M_2, & j = 1,2,\ldots \end{array}\right\} \tag{2.9}$$

holds for some constants M_1 and M_2 with $M_1^{1/p'}M_2^{1/p} = \|A\|_p$.

Now we prove that Conjecture(2.2) is still false.

Theorem 2.9 *Let $p > 1$. if Conjecture(2.2) holds for $A + E$, where $A= (a_{ij})$ is a non-negative matrix in (l_p, l_p) and either*

(i). *E is the infinite identity matrix, or*

(ii). *$E = (e_{ij})$ with $e_{ii} = 1$ for $i = 1, 2, \cdots, n$ and all other $e_{ij} = 0$, and $a_{ij} = 0$ for all $i, j > n$,*

then

$$\|A + E\|_p = \|A\|_p + 1.$$

Proof: Let $M = \|A + E\|_p$. Applying Conjecture(2.2) to $A + E$, we see that there is a positive sequence $u = (u_j)$ such that

$$\sum_{j=1}^{\infty} a_{ij}u_j^{1/p} \leqslant (M_1 - 1)u_i^{1/p}, \qquad i = 1, 2, \cdots,$$

$$\sum_{i=1}^{\infty} a_{ij}u_i^{1/p'} \leqslant (M_2 - 1)u_j^{1/p'}, \qquad j = 1, 2, \cdots,$$

with $M = M_1^{1/p'}M_2^{1/p}$. By Theorem(1.5) and Hölder's inequality, we get

$$\|A\|_p \leqslant (M_1 - 1)^{1/p'}(M_2 - 1)^{1/p} \leqslant M - 1 = \|A + E\|_p - 1 \leqslant \|A\|_p. \qquad \square$$

To show that Conjecture(2.2) is not true in general, we require, in addition to Theorem(2.9), the following proposition concerning $n \times n$ matrices which is due to Koskela [31]. It should be noted that the p-norm of an $n \times n$ matrix $A = (a_{ij})$ with respect to the l_p space of n-tuples is the same as the p-norm of the infinite form of that matrix obtained by setting $a_{ij} = 0$ for all $i, j > n$.

Proposition 2.1 *Let $p > 1$, let I denote the unit $n \times n$ matrix, and let A be a non-negative $n \times n$ matrix. Then*

$$\|A + I\|_p = \|A\|_p + 1$$

if and only if $\|A\|_p = \lambda_A$, the greatest non-negative eigenvalue of A.

Since there are non-negative $n \times n$ matrices A with greatest non-negative eigenvalues $\lambda_A < \|A\|_p$, the failure, in general, of Conjecture(2.2) follows from (ii) of Theorem(2.9) and the proposition. A simple example of such a matrix is given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

for which $\lambda_A = 0 < \|A\|_p = 1$.

Chapter 3

Subalgebras Γ_w and Ω_w of $B(X)$

3.1 Introduction

In this chapter we study subalgebras Γ_w and Ω_w of the algebra of all bounded linear operators on a non-reflexive Banach space. The algebras Γ_w and Ω_w will be defined in Definition(3.1). We first need to introduce some notation.

Given a Banach space X and $w \in X$, we denote by $\langle w \rangle$ the one dimensional subspace of X generated by w. For two subspaces S_1 and S_2, the subspace generated by them is written as $S_1 + S_2$. If $S_1 \cap S_2 = \{0\}$, we write $S_1 + S_2$ as $S_1 \oplus S_2$. If $w \notin S_1$, then we write $\langle w \rangle \oplus S_1$ as $w \oplus S_1$. For $z \in X$ and $f \in X^*$ we define a one-dimensional operator $z \otimes f \in B(X)$ by

$$(z \otimes f)(x) = f(x)z, \quad \text{for } x \in X.$$

All one-dimensional operators in $B(X)$ can be defined this way.

For Banach spaces X, Y, and $T \in B(X, Y)$, the conjugate operator $T^* \in B(Y^*, X^*)$ is defined by

$$T^*(y^*)(x) = y^*(Tx), \quad \text{for } y^* \in Y^* \text{ and } x \in X.$$

For $x \in X$ the natural imbedding of X into X^{**} is defined by $x \mapsto \hat{x}$ where $\hat{x}(x^*) = x^*(x)$ for $x^* \in X^*$. We write $\hat{X}$ as the image of X under the natural imbedding.

The Banach space X is called reflexive if $X^{**} = \hat{X}$.

It is elementary to show that

$$(z \otimes f)^{**}w = w(f)\hat{z} \quad \text{for } w \in X^{**}.$$

This formula was used in [12] to establish some general properties of the general Γ_w. The one-dimensional operators were used in the determination of the form of algebra isomorphisms between various Γ_w and Ω_w algebras. See Theorems(3.8) and (3.9).

The spaces c_0, c, l_1 and l_∞ were introduced on page 2 of Chapter 1. They are all non-reflexive BK-spaces. We also use the sequences e and e_n $(n = 1, 2, \ldots)$ and the operators P_n introduced in §1.1. For a set E of non-negative integers, we write $\chi(E)$ for the sequence $\{x_k\}$ where $x_k = 1$ for $k \in E$ and $x_k = 0$ otherwise. Accordingly, for $n = 1, 2, \ldots$, $e_n = \chi(\{n\})$. In using the sequences defined above, usually the context will make it clear as to which space the sequences are in.

The usual representations of c_0^* as l_1 and that of l_1^* as l_∞ are used in this chapter. Sometimes we make the identification of $\hat{c}_0$ with c_0 itself.

The situation that obtains when $X = c$ justifies the definitions of the algebras Γ_w and Ω_w. We take time to review the case $X = c$.

For $x = \{x_k\}_{k=1}^\infty \in c$, $\lim_k x_k$ is sometimes written as $\lim x$. Clearly $|\lim x| \leqslant \|x\|_\infty$ and $\lim e = 1$. So $\lim \in c^*$ and $\|\lim\| = 1$. By the Hahn-Banach theorem, the functional lim can be extended to l_∞ while preserving the norm 1. We call such an extension an extended limit.

For $x = \{x_k\}_{k=1}^\infty \in c$, we have

$$x = (\lim x)e + \sum_{k=1}^{\infty}(x_k - \lim x)e_k.$$

For $f \in c^*$, we have $\sum_k |f(e_k)| < \infty$ so we can define $\kappa(f) = f(e) - \sum_{k=1}^\infty f(e_k)$, and then we have

$$f(x) = \left\{ f(e) - \sum_{k=1}^{\infty} f(e_k) \right\} \lim x + \sum_{k=1}^{\infty} x_k f(e_k) = \kappa(f) \lim x + \sum_{k=1}^{\infty} x_k f(e_k). \tag{3.1}$$

It is then easy to show that

$$\|f\| = |\kappa(f)| + \sum_{k=1}^{\infty} |f(e_k)|.$$

Hence there is an isometry between c^* and l_1 given by

$$f \longleftrightarrow \{\kappa(f), f(e_1), f(e_2), \ldots\}.$$

Also $c^{**} \cong l_1^* \cong l_\infty$, according to the map

$$f \in l_1^* \longleftrightarrow \{f(e_1), f(e_2), \ldots\} \in l_\infty.$$

These identifications give $\hat{x} = \{\lim x, x_1, x_2, \ldots\}$ for $x \in c$, so that

$$\hat{c} = \{x = \{x_1, x_2, \ldots\} \in l_\infty \mid \lim x_k = x_1\}.$$

When we mention the dual and the double dual of c, this identification will be implicit. Note that in this identification $e_1 \in c^{**} \setminus \hat{c}$.

Classical summability is concerned with matrices $A = (a_{nk})$ that transform sequences $x = \{x_k\}$ according to $y_n = \sum_k a_{nk} x_k$ (see (1.1). A is called **conservative** if $A \in (c, c)$. The algebra of all conservative matrices is denoted by Γ.

For $A = (a_{nk})_{n,k=1}^{\infty} \in \Gamma$, define

$$\chi(A) = \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} - \sum_{k=1}^{\infty} \lim_{n\to\infty} a_{nk} = a - \sum_{k=1}^{\infty} a_k = \kappa(\lim \circ A).$$

where $a = \lim_n \sum_{k=1}^{\infty} a_{nk}$ and $a_k = \lim_n a_{nk}$. It is easy to see that $\sum_k |a_k| < \infty$. Then χ is a multiplicative linear functional on Γ ([14]).

For $x = \{x_k\} \in c$, by (3.1),

$$\lim Ax = \kappa(\lim \circ A) \lim x + \sum_{k=1}^{\infty} (\lim Ae_k) x_k = \chi(A) \lim x + \sum_{k=1}^{\infty} a_k x_k.$$

If $T \in B(c)$, we define

$$\chi_n(T) = \kappa(P_n \circ T), \quad \chi(T) = \kappa(\lim \circ T), \quad a_k = \lim Te_k, \quad \text{and} \quad a_{nk} = P_n Te_k.$$

Now the functional χ, which is the extension of the χ above, is not multiplicative on $B(c)$. For $x = \{x_k\} \in c$,

$$\lim Tx = \kappa(\lim \circ T)\lim x + \sum_{k=1}^{\infty}(\lim Te_k)x_k = \chi(T)\lim x + \sum_{k=1}^{\infty} a_k x_k.$$

Also for $n = 1, 2, \ldots,$

$$P_n Tx = \kappa(P_n \circ T)\lim x + \sum_{k=1}^{\infty} a_{nk}x_k = \chi_n(T)\lim x + \sum_{k=1}^{\infty} a_{nk}x_k.$$

So

$$Tx = (v \otimes \lim)x + \left\{\sum_{k=1}^{\infty} a_{nk}x_k\right\}, \tag{3.2}$$

where $v = \{v_n\} = \{\chi_n(T)\} \in l_\infty$ and $A \in (c_0, c)$. Given $T \in B(c)$, the decomposition in (3.2) is unique.

Now $T^* \in B(l_1)$ is given by the matrix:

$$T^* = \begin{pmatrix} \chi(T) & \chi_1(T) & \chi_2(T) & \ldots \\ a_1 & a_{11} & a_{21} & \ldots \\ a_2 & a_{12} & a_{22} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

and $T^{**} \in B(l_\infty)$ is given by the matrix

$$T^{**} = \begin{pmatrix} \chi(T) & a_1 & a_2 & \ldots \\ \chi_1(T) & a_{11} & a_{12} & \ldots \\ \chi_2(T) & a_{21} & a_{22} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix}.$$

It was noted in [14] that $T \in \Gamma$ if and only if $\chi_n(T) = 0$ for $n = 1, 2, \ldots$, and in this case, $T^{**}e_1 = \chi(T)e_1$.

Brown, Kerr and Stratton defined $\Omega \subset B(c)$ by $T \in \Omega$ if and only if $\{\chi_n(T)\} \in c$. Then if $T \in B(c)$ is decomposed according to (3.2), we have $A \in \Gamma$. So we may define $\rho(T) = \chi(A)$. Then ρ is a multiplicative linear functional on Ω, but Brown *et al*[14] showed that it cannot be extended to a multiplicative linear functional on $B(c)$. Indeed, they showed that $B(c)$ has no non-zero multiplicative linear functionals.

Now $T \in \Omega$ if and only if

$$T^{**}e_1 = \{\chi(T), \chi_1(T), \chi_2(T), \ldots\} = \rho(T)e_1 + \{\chi(T) - \rho(T), \chi_1(T), \chi_2(T), \ldots\}.$$

Now

$$\chi(T) - \rho(T) = \chi(T) - \chi(A) = \lim Te - \sum_{k=1}^{\infty} \lim Te_k - \lim_{n\to\infty} \sum_{k=1}^{\infty} a_{nk} + \sum_{k=1}^{\infty} a_k$$

$$= \lim Te - \lim_{n\to\infty} \sum_{k=1}^{\infty} P_n Te_k = \lim_{n\to\infty} \left(P_n Te - \sum_{k=1}^{\infty} P_n Te_k \right) = \lim_{n\to\infty} \chi_n(T)$$

Hence $T \in \Omega$ if and only if $T^{**}e_1 \in e_1 \oplus \hat{c}$.

Following Wilansky [47], but using different notation, we now give the definitions of the algebras Γ_w and Ω_w.

Definition 3.1 *Let X be a non-reflexive Banach space over $\mathbb{C}$. We define*

$$\Gamma_w = \{T \in B(X) \mid T^{**}w \in \langle w \rangle\}, \quad \textit{for } w \in X^{**}, \textit{ and}$$

$$\Omega_w = \{T \in B(X) \mid T^{**}w \in w \oplus \hat{X}\}, \quad \textit{for } w \in X^{**} \setminus \hat{X}.$$

For $w \in X^{**} \setminus \hat{X}$, we define $\rho = \rho_w : \Omega_w \to \mathbb{C}$ by the equation

$$T^{**}w = \rho_w(T)w + \hat{x}, \quad T \in \Omega_w. \tag{3.3}$$

Here the definition of Ω_w can be extended to $w \in \hat{X}$ by setting $\Omega_w = B(X)$. It is clear that for $w \in X^{**} \setminus \hat{X}$ and $T \in \Gamma_w$, $T^{**}(w) = \rho_w(T)w$.

Wilansky [47] proved the following properties of these algebras:

Theorem 3.1 *For $w \in X^{**}$, Γ_w is a closed subalgebra of $B(X)$. For $w \in X^{**} \backslash \hat{X}$, Ω_w is a closed subalgebra of $B(X)$ and ρ_w is a nonzero continuous scalar homomorphism on Ω_w.*

Here are some properties of Γ_w and Ω_w we will frequently use. The proof of them can be found in Brown, Cass, and Robinson[12].

Theorem 3.2 *Let X be a non-reflexive Banach space, and let $w_1, w_2 \in X^{**}$,*

(i). *If $w_1 = \mu w_2$ for some $\mu \neq 0$, then $\Gamma_{w_1} = \Gamma_{w_2}$. If $\langle w_1 \rangle + \hat{X} = \langle w_2 \rangle + \hat{X}$, then $\Omega_{w_1} = \Omega_{w_2}$.*

(ii). *If $w_i \neq 0$ for $i = 1, 2$, and $w_1 \notin \langle w_2 \rangle$, then $\Gamma_{w_1} \setminus \Gamma_{w_2} \neq \emptyset$ and $\Gamma_{w_2} \setminus \Gamma_{w_1} \neq \emptyset$.*

(iii). *If $w_1 \neq 0$, then $\Gamma_{w_1} \neq \Omega_{w_1}$.*

Corollary 3.3 *$\Gamma_{w_1} = \Gamma_{w_2}$ if and only if there is a number $\mu \neq 0$ such that $w_1 = \mu w_2$.*

The question of whether $w_1 \oplus \hat{X} = w_2 \oplus \hat{X}$ when $\Omega_{w_1} = \Omega_{w_2}$ and $w_1, w_2 \in X^{**} \setminus \hat{X}$ is not yet resolved in general. In [13], Brown and Cho investigated these algebras when $X = c$ and gave a positive answer as follows.

Theorem 3.4 *Let $w \in c^{**} \setminus \hat{c}$, then $\Omega_z = \Omega_w$ if and only if $z \in (w \oplus \hat{c}) \setminus \hat{c}$.*

In addition to Theorem(3.4), Brown and Cho studied isomorphisms between subalgebras Γ_w and Ω_w of $B(c)$ in [13]. They proved, *inter alia*, the following theorem:

Theorem 3.5 *Let $w \in c^{**} \setminus \hat{c}$, then there exists an automorphism T in $B(c)$ such that $T^{**}w = e_1 \in c^{**} \setminus \hat{c}$. Hence for $w, z \in c^{**} \setminus \hat{c}$, $\Gamma_w \cong \Gamma_z$ and $\Omega_w \cong \Omega_z$.*

Brown, Cass and Robinson[12], and Cass[15], [16] continued the study of subalgebras Γ_w and Ω_w. They proved some results concerning isomorphisms between, and intersections of these algebras in a general non-reflexive Banach space X. They also showed that when $X = l_1$, the relationships between these algebras are more complicated than in c. The following are some of the results. Here and later we use I to denote the identity operator.

Theorem 3.6 **([12, Theorem 5])**

$$\bigcap\{\Omega_w \mid w \in X^{**} \setminus \hat{X}\} = \langle I \rangle \oplus W,$$

where W is the set of all weakly compact operators in $B(X)$.

Theorem 3.7 **([12, Theorem 6])**

$$\bigcap\{\Gamma_w \mid w \in X^{**}\} = \langle I \rangle.$$

Theorem 3.8 **([12, Theorem 9])** *Let X be an non-reflexive Banach space and let $w, z \in X^{**} \setminus \hat{X}$. If $\varphi : \Gamma_w \to \Gamma_z$ is an algebra isomorphism, then there is a linear homeomorphism $T \in B(X)$ such that*

$$\varphi(U) = TUT^{-1} \qquad \text{for } U \in \Gamma_w. \tag{3.4}$$

Theorem 3.9 **([15, Theorem 1])** *Let X be an non-reflexive Banach space and let $w, z \in X^{**} \setminus \hat{X}$. If $\varphi : \Omega_w \to \Omega_z$ is an algebra isomorphism, then there is a linear homeomorphism $T \in B(X)$ such that*

$$\varphi(U) = TUT^{-1} \qquad \text{for } U \in \Omega_w. \tag{3.5}$$

Using Theorems(3.8) and (3.9), the following theorems were proved by a cardinality argument.

Theorem 3.10 ([12, Theorem 10]) *There are points $w, z \in l_1^{**} \setminus \hat{l}_1$ such that Γ_w is not isomorphic to Γ_z.*

Theorem 3.11 ([15, Theorem 2]) *There are points $w, z \in l_1^{**} \setminus \hat{l}_1$ such that Ω_w is not isomorphic to Ω_z.*

In Cass[16] it was shown that if $z \in X$ and $w \in X^{**} \setminus \hat{X}$, then $\Gamma_{\hat{z}} \not\cong \Gamma_w$ and if $z_1, z_2 \in X$, then $\Gamma_{\hat{z}_1} \cong \Gamma_{\hat{z}_2}$.

In the remainder of this chapter we continue the study of the subalgebras Γ_w and Ω_w. In §3.2 we show that if $X = c_0$, then $\Omega_w \cong \Omega_z$ and $\Gamma_w \cong \Gamma_z$ for $w, z \in c_0^{**} \setminus \hat{c}_0$. In §3.3, we will explore isomorphisms between subalgebras associated with two kinds of points in $l_1^{**} \setminus \hat{l}_1$, the Dirac measures and Banach limits (defined in Definition(3.3)). In §3.4, we will seek explicit representations of intersections of subalgebras associated with the above two kinds of points. Results in this chapter can be found in [20].

3.2 Subalgebras of $B(c_0)$

To establish a result similar to Theorem(3.5) for $B(c_0)$, we first prove the following lemma.

Lemma 3.12 *Let $w \in l_\infty \setminus c_0$, then there is an invertible $T \in B(c_0)$ such that $T^{**}w$ is a sequence of zeros and ones.*

PROOF: Suppose $w = \{w_1, w_2, \ldots\} \in l_\infty \setminus c_0$, and let $\limsup_{n\to\infty} |w_n| = \delta > 0$. Let $\{w_{n_k}\}$ be a subsequence of w such that $|w_{n_k}| \geqslant \delta/2$ for all k and $\lim_{k\to\infty} |w_{n_k}| = \delta$. Write $n_0 = 0$ and let $w_{n'_k}$ be the first term of w having the largest absolute value among $\{w_{n_{k-1}+1}, w_{n_{k-1}+2}, \ldots, w_{n_k}\}$.

Let $T_k = (a_{ij}^{(k)})$ be an $(n_k - n_{k-1}) \times (n_k - n_{k-1})$ matrix with $a_{ij}^{(k)}$ defined as:

$$a_{ij}^{(k)} = \begin{cases} 1/w_{n'_k} & \text{for } i = j = n'_k - n_{k-1}; \\ -w_{n_{k-1}+i}/w_{n'_k} & \text{for } j = n'_k - n_{k-1}, i \neq j; \\ 1 & \text{for } i = j \neq n'_k - n_{k-1}; \\ 0 & \text{otherwise.} \end{cases}$$

In matrix form we have

$$T_k = \begin{pmatrix} 1 & 0 & \cdots & -w_{n_{k-1}+1}/w_{n'_k} & \cdots & 0 \\ 0 & 1 & \cdots & -w_{n_{k-1}+2}/w_{n'_k} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1/w_{n'_k} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -w_{n_k}/w_{n'_k} & \cdots & 1 \end{pmatrix};$$

and

$$T_k^{-1} = \begin{pmatrix} 1 & 0 & \cdots & w_{n_{k-1}+1} & \cdots & 0 \\ 0 & 1 & \cdots & w_{n_{k-1}+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & w_{n'_k} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{n_k} & \cdots & 1 \end{pmatrix}$$

Note that $\|T_k\|_\infty \leqslant \max\{2, 2/\delta\}$ and $\|T_k^{-1}\|_\infty \leqslant 1 + \|w\|_\infty$ for $k = 1, 2, \ldots$.

Let

$$T = \begin{pmatrix} T_1 & 0 & 0 & \cdots \\ 0 & T_2 & 0 & \cdots \\ 0 & 0 & T_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

then $T \in B(c_0)$ and is invertible, with $T^{**}w$ being a sequence of zeros and ones. □

Theorem 3.13 *Let $w \in l_\infty \setminus c_0$. Then*

(i) *there is an invertible $S \in B(c_0)$ such that $S^{**}w = e$,*

(ii) *for $z \in l_\infty \setminus c_0$, $\Gamma_w \cong \Gamma_z$ and $\Omega_w \cong \Omega_z$.*

PROOF: (i) By Lemma(3.12), we may assume that w is a sequence of zeros and ones with infinitely many ones, say $w = \chi(\{n_k\})$. Let

$$S_k = \begin{pmatrix} 1 & 0 & \cdots & 1 \\ 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

which is of order $(n_k - n_{k-1}) \times (n_k - n_{k-1})$ with $n_0 = 0$. We see that S_k is invertible, $\|S_k\|_\infty \leqslant 2$ and $\|S_k^{-1}\|_\infty \leqslant 2$. Now

$$S = \begin{pmatrix} S_1 & 0 & 0 & \cdots \\ 0 & S_2 & 0 & \cdots \\ 0 & 0 & S_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

is the matrix needed to map w onto e.

(ii) This follows directly from (i) and Theorems(3.8) and (3.9). □

Remark. By definition, $T \in B(c_0)$ is in Ω_e if and only if $T^{**}e = \rho e + \hat{x}_0$ for some $\rho \in \mathbb{C}$ and $x_0 \in c_0$. This is equivalent to saying that for all $x \in c$, $T^{**}x = \rho(\lim x)e + \widehat{x_0'}$ for some $x_0' \in c_0$. Thus, $\lim T^{**}x = \rho \lim x$ with ρ independent of $x \in c$. Hence, if we consider operators in $B(c_0)$ as matrices, then Ω_e is exactly the collection of all multiplicative conservative matrices as defined in [14]. Thus, Theorem(3.13) shows that for each $w \in l_\infty \setminus c_0$, Ω_w is isomorphic to the algebra of all multiplicative conservative matrices.

The following result for c_0 is similar to Theorem(3.4).

Theorem 3.14 *Let $w = \{w_n\}$ and $z = \{z_n\} \in l_\infty$.*

(i). *If $z \notin \hat{c}_0$ and $w \notin z \oplus \hat{c}_0$, then neither Ω_w nor Ω_z contains the other.*

(ii). *$\Omega_z = \Omega_w$ if and only if $\langle w\rangle + \hat{c}_0 = \langle z\rangle + \hat{c}_0$.*

PROOF: (i) We first show that we can always find two disjoint increasing sequences $\{n_k'\}$ and $\{n_k''\}$ of natural numbers, such that all of $w' = \lim_k w_{n_k'}$, $w'' = \lim_k w_{n_k''}$, $z' = \lim_k z_{n_k'}$ and $z'' = \lim_k z_{n_k''}$ exist and $w'z'' \neq w''z'$. Because $z \notin \hat{c}_0$, we can find

$\{n'_k\}$ such that $z' = \lim_k z_{n'_k} \neq 0$. By picking a subsequence of $\{n'_k\}$ if necessary, we can assume that $\{w_{n'_k}\}$ also converges, say to w'. Because $w - (w'/z')z \notin \hat{c}_0$, there exists $\{n^*_k\}$, disjoint from $\{n'_k\}$, such that $w_{n^*_k} - (w'/z')z_{n^*_k} \to r \neq 0$. A subsequence $\{n''_k\}$ of $\{n^*_k\}$ on which both w and z converge is all we need.

Now define $T \in B(c_0)$ by

$$P_n T x = \left\{ \begin{array}{ll} (w'' - w')x_{n''_k} + w'' x_{n'_k} & \text{for } n = n''_k, k = 1, 2, \ldots, \\ w'' x_n & \text{for all other } n, \end{array} \right\}$$

where $x = \{x_n\} \in c_0$. Now $P_{n'_k}(T^{**}z - w''z) = 0$ for each k. Keeping in mind that $z' \neq 0$, we have $P_{n'_k}(T^{**}z - \rho z) \not\to 0$ for any $\rho \neq w''$. But

$$P_{n''_k}(T^{**}z - w''z) = (w'' - w')z_{n''_k} + w'' z_{n'_k} - w'' z_{n''_k} \to w''z' - w'z'' \neq 0$$

as $k \to \infty$. Hence $T \notin \Omega_z$. On the other hand, $T^{**}w - w''w \in \hat{c}_0$. Consequently $T \in \Omega_w$ and thus $T \in \Omega_w \setminus \Omega_z$.

As the conditions are in fact symmetric on z and w, we can similarly construct an operator in $\Omega_z \setminus \Omega_w$. Thus neither Ω_w nor Ω_z contains the other.

(ii) That $\Omega_z = \Omega_w$ when $\langle w \rangle + \hat{c}_0 = \langle z \rangle + \hat{c}_0$ follows from Theorem(3.2).

In light of (i), to show the reverse implication, we have only to consider the case that exactly one of w or z is in $\hat{c}_0$, say $z \in \hat{c}_0$. By definition $\Omega_z = B(c_0)$. So we have only to find an operator in $B(c_0)$ which is not in Ω_w. Suppose a subsequence $\{w_{n_k}\}$ of w satisfies $\lim_k w_{n_k} = \delta \neq 0$. Let $d_{n_{2k}} = 1$ and $d_n = 0$ otherwise. Then $D = \text{diag}\,\{d_n\} \notin \Omega_w$. □

3.3 Isomorphisms between Subalgebras of $B(l_1)$

In order to study subalgebras of $B(l_1)$, we need properties of $l_1^{**} = l_\infty^*$. We give some notation and terminology that we use in the rest of this chapter.

Let $\mathbf{N}$ be the set of all positive integers, $\beta\mathbf{N}$ the Stone-Čech compactification of $\mathbf{N}$, and $\mathbf{N}^* = \beta\mathbf{N} \setminus \mathbf{N}$. Each $x \in l_\infty$, regarded as a bounded continuous function on $\mathbf{N}$, can be extended uniquely to a continuous function on $\beta\mathbf{N}$, which we write again as x. In this way we identify l_∞ with $C(\beta\mathbf{N})$. By the Riesz Representation Theorem (see, e.g., [26, p.130]), each $f \in l_\infty^* = l_1^{**}$ corresponds to a unique regular Borel measure μ_f on $\beta\mathbf{N}$ such that $\|f\| = \|\mu_f\|$ and

$$f(x) = \int_{\beta\mathbf{N}} x \, d\mu_f \qquad \text{for } x \in l_\infty.$$

Sometimes we use f and μ_f interchangeably, for example, we could write $\mu_f(x)$ for $f(x)$. We denote by $M(\beta\mathbf{N})$ the space of all regular Borel measures on $\beta\mathbf{N}$. For $y = \{y_k\} \in l_1$, $\hat{y} \in l_1^{**}$ corresponds to the measure $\mu_y \in M(\beta\mathbf{N})$ where $\mu_y(A) = 0$ if $A \subset \mathbf{N}^*$ and $\mu_y(\{k\}) = y_k$ for $k \in \mathbf{N}$. We will also use y, $\hat{y}$ and μ_y interchangeably from time to time.

We give a brief description of constructing $\beta\mathbf{N}$ by using ultrafilters of $\mathbf{N}$, as was done in [47].

Definition 3.2 *A family p of non-empty subsets of $\mathbf{N}$ is called an* **ultrafilter** *if the following conditions are satisfied.*

(i). *If $E_1 \in p$ and $E_2 \in p$ then $E_1 \cap E_2 \in p$.*

(ii). *The family p is maximal with respect to* (i).

Fix any point $n \in \mathbf{N}$ and let $p\{n\}$ be the collection of all subsets of $\mathbf{N}$ containing $\{n\}$, then $p\{n\}$ is an ultrafilter. Such an ultrafilter is called **fixed**. All other ultrafilters are called **free**.

The following are some properties of ultrafilters on $\mathbf{N}$. Some of them are self-evident. For those that need to be proved, one can see [40].

(a). For a collection $\mathcal{F}$ of subsets of $\mathbf{N}$ satisfying (i) of Definition(3.2), there exists an ultrafilter p such that $\mathcal{F} \subset p$.

(b). If p is an ultrafilter, $E_1 \in p$ and $E_1 \subset E_2 \subset \mathbf{N}$, then $E_2 \in p$.

(c). Suppose p is an ultrafilter and $E \subset \mathbf{N}$. If for all $F \in p$ we have $E \cap F \neq \emptyset$, then $E \in p$.

(d). Suppose p is an ultrafilter and $D \in p$. For $E \subset D$, we have either $E \in p$ or $D \setminus E \in p$. In particular, for any $E \subset \mathbf{N}$, either $E \in p$ or $\mathbf{N} \setminus E \in p$.

(e). Given any two ultrafilters p_1 and p_2. If $p_1 \neq p_2$, then there exists $E \subset \mathbf{N}$ such that $E \in p_1$ and $\mathbf{N} \setminus E \in p_2$.

(f). An ultrafilter p is fixed if and only if it contains a finite subset of $\mathbf{N}$.

(g). Suppose p is a free ultrafilter and E, F are subsets of $\mathbf{N}$, where F is finite. Then $E \in p$ if and only if $E \setminus F \in p$.

We can give a nice description of the topology on $\beta\mathbf{N}$ using the following idea given in Rudin[40]. Let S be the collection of all ultrafilters on $\mathbf{N}$. For every subset E of $\mathbf{N}$, let $V(E)$ be the set of all ultrafilters p such that $E \in p$. Then for $E_1, E_2 \subset \mathbf{N}$, we have

(i). $V(E_1 \cap E_2) = V(E_1) \cap V(E_2)$,

(ii). $V(\mathbf{N} \setminus E) = S \setminus V(E)$,

(iii). $V(E_1 \cup E_2) = V(E_1) \cup V(E_2)$.

So, we can define a topology on S by declaring each of the sets $V(E)$ open. We can observe from the above properties that for all $E \subset \mathbf{N}$, $V(E) = \overline{E}$ (here and elsewhere $\overline{E}$ stands for the closure of E in S).

Theorem 3.15 (Rudin [40]) *S is homeomorphic to $\beta\mathbf{N}$.*

In the remainder of this chapter we will identify $\beta\mathbf{N}$ with S as given above. In this identification each point $n \in \mathbf{N}$ corresponds to the fixed ultrafilter $p\{n\}$.

Let $p \in \mathbf{N}^*$. We use the notation Ω_p and Γ_p, respectively, for the subalgebras Ω_{δ_p} and Γ_{δ_p}, where δ_p is the Dirac measure at p. Cass showed in [15] that if p and q are different, then $\Omega_p \neq \Omega_q$. For $x = \{x_k\}_{k=1}^{\infty} \in l_\infty$, the translation Tx of x is defined by $Tx = \{x_{n+1}\}_{n=1}^{\infty}$.

Definition 3.3 *A linear functional f on l_∞ is called a* **Banach limit** *if it satisfies:*

(i). $f(x) = \lim x$ *for all* $x = \{x_n\} \in c$;

(ii). $\|f\| = 1$;

(iii). f *is translation invariant, i.e.,* $f(x) = f(Tx)$ *for all* $x \in l_\infty$.

The set of all Banach limits is denoted by BL.

Thus a Banach limit is an extended limit which is translation invariant. Properties (i) and (ii) above imply:

(iv). If $x = \{x_n\} \in l_\infty$ with $x_n \geqslant 0$, then $f(x) \geqslant 0$. Hence if $x = \{x_n\} \in l_\infty$ and $x_n \in \mathbf{R}$ for all n, then $\liminf_n x_n \leqslant f(x) \leqslant \limsup_n x_n$.

Proof of (iv): Take $x = \{x_n\}$, with $x_n \geqslant 0$ for all $n \in \mathbf{N}$. From (i), we have $f(x) + f(\|x\|e - x) = f(\|x\|e) = \|x\|$. Since $x_n \geqslant 0$ $\forall n$, $\|(\|x\|e - x)\| \leqslant \|x\|$, so by (ii) we have $\|x\| \geqslant f(\|x\|e - x) = \|x\| - f(x)$, which implies $f(x) \geqslant 0$. □

Remark. Unlike the Dirac measures, the measures on $\beta\mathbf{N}$ associated with Banach limits are continuous. Indeed, consider any such measure μ and any $p \in \beta\mathbf{N}$. For

$n \in \mathbf{N}$, let $E_{i,n} = \{k \in \mathbf{N} \mid k \equiv i \bmod (n)\}$ where $0 \leqslant i \leqslant n-1$. Then $p \in \overline{E}_{i,n}$ for some i and $\mu(\{p\}) \leqslant \mu(\overline{E}_{i,n}) = 1/n$. Hence $\mu(\{p\}) = 0$.

Theorems(3.8) and (3.9) were proved using cardinality arguments, so they fail to produce any concrete examples of subalgebras that are not isomorphic. A[illegible]ubalgebras of two Dirac measures can be non-isomorphic, it might be plausible to expect that those of two quite different measures, like a Dirac measure and a Banach limit, cannot be isomorphic to each other. But, we show in Theorem(3.17) that the subalgebras associated with a Dirac measure are isomorphic to those associated with some Banach limit. First, we state a lemma ([2, Lemma 1]) that gives criteria for a functional $f \in l_\infty^*$ to be a Banach limit.

Lemma 3.16 *Suppose f is a continuous linear functional on l_∞, then f is a Banach limit if and only if the following three properties are satisfied:*

(i). $\|f\| = 1$;

(ii). $f(e) = 1$;

(iii). $f(x) = 0$ *for all* $x \in bs$.

Here bs is the space of series with bounded partial sums, i.e.,

$$bs = \left\{ x = \{x_k\}_{k=1}^{\infty} \;\middle|\; \|x\|_{bs} = \sup_{n \geqslant 1} \left| \sum_{k=1}^{n} x_k \right| < \infty \right\} \tag{3.6}$$

Proposition 3.1

$$\bigcap\{\Gamma_f : f \in BL\} \setminus \bigcup\{\Omega_p : p \in \mathbf{N}^*\} \neq \emptyset, \tag{3.7}$$

and

$$\bigcap\{\Gamma_p : p \in \mathbf{N}^*\} \setminus \bigcup\{\Omega_f : f \in BL\} \neq \emptyset. \tag{3.8}$$

PROOF: Define $A = (a_{nk}) \in B(l_1)$ (the unilateral shift operator) by $a_{n+1,n} = 1$ for $n = 1, 2, \ldots$ and $a_{nk} = 0$ otherwise. $T = A^*$ is the translation operator on l_∞, cited in Definition(3.3). For every Banach limit f, $A^{**}f = f$, so $A \in \bigcap\{\Gamma_f : f \in BL\}$.

On the other hand, if $p \in \mathbf{N}^*$, then $A \notin \Omega_p$. Indeed, assume the contrary, and suppose $A^{**}\delta_p = \rho\delta_p + \hat{y}$, where $y = \{y_k\} \in l_1$. Then for $x = \{x_k\} \in l_\infty$, we have

$$\int_{\beta\mathbf{N}} Tx\, d\delta_p = (A^{**}\delta_p)(x) = \rho \int_{\beta\mathbf{N}} x\, d\delta_p + \sum_{k=1}^{\infty} y_k x_k.$$

By putting $x = e_k$ we see $y_k = 0$. Now, for $E = \{2n\}_{n=1}^{\infty}$, p is either in $\overline{E}$ or $\overline{\mathbf{N} \setminus E}$. Suppose without loss of generality that $p \in \overline{E}$. Putting $x = \chi(\mathbf{N} \setminus E)$ yields: $1 = \int_{\beta\mathbf{N}} Tx\, d\delta_p = \rho \int_{\beta\mathbf{N}} x\, d\delta_p = \rho \cdot 0 = 0$, a contradiction which shows that $A \notin \Omega_p$. Thus (3.7) holds.

For (3.8), define $D = (d_{nk}) \in B(l_1)$ by putting $d_{2m,2m} = 1$ for all $m \in \mathbf{N}$ and $d_{nk} = 0$ otherwise (D is the "even-term picking" operator). Then, for $p \in \mathbf{N}^*$ and $E = \{2n\}$, depending on whether $p \in \overline{E}$ or $p \in \overline{\mathbf{N} \setminus E}$, we have either $D^{**}\delta_p = \delta_p$ or $D^{**}\delta_p = 0$. In both cases we have $D \in \Gamma_p$. But, if there is a $\rho \in \mathbf{C}$ and $z = \{z_k\} \in l_1$ such that $D^{**}f = \rho f + \hat{z}$ for some Banach limit f, then, with $x = e_k$ we see $z_k = 0$. Again, putting $x = e$ we find $\rho = 1/2$. But, if $x = \chi(E)$ we find $\rho = 1$, a contradiction which shows $D \notin \Omega_f$. Hence (3.8) is proved. □

In order to establish isomorphisms between the subalgebras we are investigating, we construct linear homeomorphisms in $B(l_1)$. So we need to clarify some relationships between permutations of $\mathbf{N}$ and linear homeomorphisms of l_1, l_∞, and l_1^{**}. Any permutation u of $\mathbf{N}$ (i.e., a one-to-one mapping from $\mathbf{N}$ onto itself) can be extended to a unique homeomorphism $\hat{u}$ of $\beta\mathbf{N}$. A linear isometry $\tilde{u}$ of $M(\beta\mathbf{N})$ can then be defined as follows. For a regular Borel measure μ on $\beta\mathbf{N}$, we define

$$\tilde{u}(\mu)(A) = \mu(\hat{u}^{-1}(A))$$

for a Borel subset A of $\beta\mathbf{N}$. Clearly $\tilde{u}(\mu) \in M(\beta\mathbf{N})$.

On the other hand, for the same u, a bounded linear operator $U \in B(l_1)$ is defined by $Ue_j = e_{u(j)}$ for $j \in \mathbf{N}$, so $U \in B(l_1)$ is given by the matrix $U = \{u_{nk}\}$, where

$$u_{nk} = \begin{cases} 1 & \text{if } u(k) = n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

This means that for a sequence $x = \{x_i\} \in l_\infty$, $Ux = \{x_{u^{-1}(i)}\}$ and $U^* \in B(l_\infty)$ is given by the transpose of the matrix U.

Now U^{**} is a linear isometry of $l_\infty^* = M(\beta\mathbf{N})$. In fact, $U^{**} = \tilde{u}$ because for $\mu \in M(\beta\mathbf{N})$, and $x = \{x_k\} \in l_\infty$, we have

$$(U^{**}\mu)(x) = \mu(U^*x) = \int_{\beta\mathbf{N}} U^*\{x_k\}\, d\mu = \int_{\beta\mathbf{N}} \{x_{u(k)}\}\, d\mu = \int_{\beta\mathbf{N}} \{x_n\}\, d\tilde{u}(\mu) = \tilde{u}(\mu)(x).$$

Theorem 3.17 *For each $p \in \mathbf{N}^*$, there is an invertible $T \in B(l_1)$ such that $T^{**}\delta_p$ is a Banach limit.*

PROOF: For $n \geqslant 3$, let $R_n = (r_{ij})$ be the following $n \times n$ matrix:

$$\begin{pmatrix} 1/n & 1 & 1 & 1 & \cdots & 1 \\ 1/n & 1 & 0 & 0 & \cdots & 0 \\ 1/n & 0 & 1 & 0 & \cdots & 0 \\ 1/n & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/n & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Now $\sup_{1 \leqslant j \leqslant n} \sum_{i=1}^{n} |r_{ij}| = 2$, and the inverse of R_n is:

$$R_n^{-1} = (s_{ij})_{n\times n} = \begin{pmatrix} -\frac{n}{n-2} & \frac{n}{n-2} & \frac{n}{n-2} & \frac{n}{n-2} & \cdots & \frac{n}{n-2} \\ \frac{1}{n-2} & 1-\frac{1}{n-2} & -\frac{1}{n-2} & -\frac{1}{n-2} & \cdots & -\frac{1}{n-2} \\ \frac{1}{n-2} & -\frac{1}{n-2} & 1-\frac{1}{n-2} & -\frac{1}{n-2} & \cdots & -\frac{1}{n-2} \\ \frac{1}{n-2} & -\frac{1}{n-2} & -\frac{1}{n-2} & 1-\frac{1}{n-2} & \cdots & -\frac{1}{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{n-2} & -\frac{1}{n-2} & -\frac{1}{n-2} & -\frac{1}{n-2} & \cdots & 1-\frac{1}{n-2} \end{pmatrix},$$

which satisfies $\sup_{1 \leqslant j \leqslant n} \sum_{i=1}^{n} |s_{ij}| \leqslant 4$. Now define the infinite matrices

$$R = \begin{pmatrix} R_3 & 0 & 0 & \cdots \\ 0 & R_4 & 0 & \cdots \\ 0 & 0 & R_5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \text{ and } S = \begin{pmatrix} R_3^{-1} & 0 & 0 & \cdots \\ 0 & R_4^{-1} & 0 & \cdots \\ 0 & 0 & R_5^{-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We see that both R and S are in $B(l_1)$, and that $S = R^{-1}$.

Let

$$n_k = \frac{(k+1)(k+2)}{2} - 2 = 1 + \sum_{i=1}^{k-1}(i+2),$$

and define $E = \{n_k \mid k \in \mathbf{N}\}$. For any $p \in \overline{E} \setminus E$, we claim that $R^{**}\delta_p$ is a Banach limit. To see this, we first observe that for $x = \{x_n\} \in l_\infty$, the n_k-th term of R^*x is $P_{n_k}R^*x = \sum_{i=0}^{k+1} x_{n_k+i}/(k+2)$, and $(R^{**}\delta_p)(x) = \int_{\beta\mathbf{N}} R^*x\, d\delta_p$ is a subsequential limit of the sequence $\{P_{n_k}R^*x\}$. Since we also have $(R^{**}\delta_p)(e) = 1$, we see that $\|P^{**}\delta_p\| = 1$. For any $x = \{x_n\} \in bs$, $P_{n_k}R^*x = \sum_{i=0}^{k+1} x_{n_k+i}/(k+2) \to 0$ as $k \to \infty$, so $(R^{**}\delta_p)(x) = 0$. By Lemma(3.16), $R^{**}\delta_p$ is a Banach limit. Putting $T = R$ yields the result for $p \in \overline{E} \setminus E$.

For an arbitrary $p \in \mathbf{N}^*$, take an infinite subset $P \subset \mathbf{N}$ with infinite complement in $\mathbf{N}$ such that $p \in \overline{P}$. Arrange the elements in each of P, $\mathbf{N} \setminus P$ and $\mathbf{N} \setminus E$ respectively in increasing order $p_1 < p_2 < \cdots$, $q_1 < q_2 < \cdots$ and $m_1 < m_2 < \cdots$, and define the matrix $U = (u_{nk})$ by $u_{n_ip_i} = u_{m_iq_i} = 1$ for $i = 1, 2, \ldots$, and $u_{nk} = 0$ otherwise. Then $U \in B(l_1)$ and its inverse $V = (v_{nk}) \in B(l_1)$ is given by $v_{p_in_i} = v_{q_im_i} = 1$ for $k = 1, 2, \ldots$, and $v_{nk} = 0$ otherwise.

As in the discussion immediately before this theorem, U is associated with a permutation u of $\mathbf{N}$ that maps P onto E, and $U^{**} = \tilde{u}$ maps a Dirac measure supported on $\overline{P}$ to a Dirac measure supported on $\overline{E}$. In particular, $U^{**}\delta_p$ is a Dirac measure supported on $\overline{E}$, and is not in $\hat{l}_1$. So $U^{**}\delta_p = \delta_q$ where $q \in \overline{E} \setminus E$. Let $T = RU$, and the result follows. □

Using Theorems(3.8) and (3.9) we get:

Corollary 3.18 *Suppose $p \in \mathbf{N}^*$, then there is a Banach limit f such that $\Gamma_p \cong \Gamma_f$ and $\Omega_p \cong \Omega_f$.* □

3.4 Intersections of Subalgebras of $B(l_1)$

We now characterize $\bigcap_{w\in L} \Gamma_w$ and $\bigcap_{w\in L}\Omega_w$ where L is either the set of Banach limits or the set of Dirac measures. Results in this section can be compared to Theorems(3.6) and (3.7), where $\bigcap\{\Omega_w \mid w \in X^{**}\setminus \hat{X}\}$ and $\bigcap\{\Gamma_w \mid w \in X^{**}\}$ were characterized.

We first establish a property of all non-negative measures μ such that $\mu(\mathbf{N}) = 0$.

Lemma 3.19 *Let $\mu \in M(\beta\mathbf{N})$ be a non-negative Borel measure such that $\mu(\mathbf{N}) = 0$. There are strictly increasing sequences of positive integers $\{n_i\}$ and $\{p_i\}$, where $n_{i+1} \geqslant n_i + p_i$ for $i = 1, 2, \ldots$, such that $\mu(\overline{E}) = 0$, where*

$$E = \bigcup_{i=1}^{\infty}\{n_i, n_i + 1, \ldots, n_i + p_i - 1\} \tag{3.9}$$

PROOF: Let $E_0 = \mathbf{N} = \bigcup_{i=1}^{\infty} E_{i,0}$ where $E_{i,0} = \{i^2, i^2+1, \ldots, i^2+2i\}$. Let

$$E_1' = \bigcup_{i=1}^{\infty} E_{2i-1,0} \text{ and } E_1'' = \bigcup_{i=1}^{\infty} E_{2i,0}.$$

So $E_0 = E_1' \cup E_1''$ and $E_1' \cap E_1'' = \emptyset$. Since disjoint subsets of $\mathbf{N}$ have disjoint closures in $\beta\mathbf{N}$ we see that

$$\mu(\overline{E_0}) = \mu(\overline{E_1'}) + \mu(\overline{E_1''}).$$

So we can choose E_1 to be either E_1' or E_1'' so that $\mu(\overline{E_1}) \leqslant (1/2)\mu(\overline{E_0})$. We similarly decompose E_1 so that $E_1 = E_2' \cup E_2''$ with $E_2' \cap E_2'' = \emptyset$ and choose E_2 such that $\mu(\overline{E_2}) \leqslant (1/2)\mu(\overline{E_1})$.

Continuing in this way we obtain a sequence of sets $\{E_n\}$ where $E_n = \bigcup_{i=1}^{\infty} E_{i,n}$, $\{E_{i,n+1}\}_i$ is a subsequence of $\{E_{i,n}\}$, and $\mu(\overline{E_n}) \to 0$ as $n \to \infty$. Let

$$E = \bigcup_{i=1}^{\infty} E_{i,i} = \bigcup_{i=1}^{\infty}\{n_i, n_i + 1, \ldots, n_i + p_i - 1\}.$$

Clearly, $\{n_i\}$ and $\{p_i\}$ are strictly increasing.

For each n, $E \setminus E_n$ is a finite subset of $\mathbf{N}$. Hence $\mu(\overline{E}) \leqslant \mu(\overline{E_n})$ so that $\mu(\overline{E}) = 0$.

□

Using Lemma(3.19), we can characterize the operators in $\bigcap\{\Gamma_f \mid f \in BL\}$.

Theorem 3.20 *An operator $T \in B(l_1)$ is in Γ_f for all $f \in BL$ if and only if there is a complex number ρ and a matrix $T_0 \in B(l_1)$ with $T_0^* \in B(l_\infty, ac_0)$, such that $T = \rho I + T_0$. Here ac_0 is the space of sequences that are almost convergent to 0; that is,*

$$ac_0 = \{x = \{x_n\} \in l_\infty \mid f(x) = 0 \textit{ for all } f \in BL\}.$$

Remark. The matrices mapping l_∞ into ac_0, or ac, are characterized in [27].

PROOF: The "if" part of the proof follows directly from the definition of Γ_f. For the "only if" part, take any $T \in \bigcap\{\Gamma_f \mid f \in BL\}$. For $f \in BL$ we have $(T^{**}f)(x) = \rho_f f(x)$ for some $\rho_f \in \mathbf{C}$ and all $x \in l_\infty$. Now ρ_f is independent of f. Indeed, take any two Banach limits a and b. By Lemma(3.19), there is a subset E of $\mathbf{N}$, given by (3.9) such that $(a+b)(\overline{E}) = 0$. Define $A : l_\infty \to l_\infty$ by $P_i Ax = (1/p_i)\sum_{k=n_i}^{n_i+p_i-1} x_k$ for $x = \{x_k\} \in l_\infty$. Take any extended limit α on l_∞ and let $h = \alpha A$. Then, since $p_i \to \infty$ so that $Ax \in c_0$ when $x \in bs$, we see that $h(x) = 0$ when $x \in bs$. Lemma(3.16) shows that $h \in BL$. Also the support of μ_h is contained in $\overline{E}$. Now for any $x \in l_\infty$, remembering that $F = (1/2)(a+h) \in BL$, we have

$$(a+h)(T^*x) = \rho_F (a+h)(x),$$

$$a(T^*x) = \rho_a a(x),$$

$$h(T^*x) = \rho_h h(x).$$

Thus $\rho_F(a+h)(x) = \rho_a a(x) + \rho_h h(x)$. Knowing that the supports of μ_a and μ_h are disjoint, we see that $\rho_F = \rho_a = \rho_h$. Similarly $\rho_b = \rho_h$. Thus $\rho_a = \rho_b$. Let $T_0 = T - \rho I$ where ρ is the common value of ρ_f, It is immediate that $T_0^* \in (l_\infty, ac_0)$. $\square$

Theorem 3.21 *If $T \in \bigcap_{f\in BL}\Omega_f$, then there is a $\rho \in \mathbb{C}$ and a weak-star to norm continuous function $\phi : BL \to l_1$ such that*

$$\phi(\lambda f + (1-\lambda)g) = \lambda\phi(f) + (1-\lambda)\phi(g) \quad \textit{for all } \lambda \in [0,1] \textit{ and } f, g \in BL. \tag{3.10}$$

Also $T^{**}f = \rho f + \widehat{\phi(f)}$ for all $f \in BL$. Moreover, $\phi(BL)$ is norm compact in l_1.

PROOF: Suppose $T \in \bigcap_{f\in BL}\Omega_f$. Then for $f \in BL$, there exist $\rho_f \in \mathbb{C}$ and $x_f = \{x_k\} \in l_1$ such that $T^{**}f = \rho_f f + \widehat{x_f}$. We define $\phi(f) = x_f$. It is clear that ϕ satisfies (3.10). Similarly as in the proof of Theorem(3.20), we see that ρ_f is independent of f.

Now BL is a w^*-compact convex subset of l_∞^*. For the weak-star to norm continuity, we pick a net $\{f_\lambda \mid \lambda \in \Lambda\}$ in BL which converges to $f \in BL$ in the w^*-topology. Then for any $z \in l_\infty$,

$$z(\phi(f_\lambda)) = f_\lambda\left[(T^* - \rho I)(z)\right] \to f\left[(T^* - \rho I)(z)\right] = z(\phi(f)),$$

so ϕ is weak-star to weak continuous. As a result, $\phi(BL)$ is weakly compact. Eberlein-Šmulian's Theorem and Schur's Theorem together give us the norm compactness of $\phi(BL)$. By a standard result in general topology, the norm and the weak topologies on $\phi(BL)$ are the same. Thus ϕ is w^*-norm continuous. $\square$

Remark. In the light of Theorem(3.20), one may be tempted to conjecture that $T \in \bigcap_{f\in BL}\Omega_f$ if and only if $T^* \in \langle I\rangle + (l_\infty, ac)$, where

$$ac = \{x \in l_\infty \mid f(x) = g(x) \text{ for all } f, g \in BL\}$$

is the space of all almost-convergent sequences. But in fact this is not true as the following example shows. Let $z = \{z_k\}$ be a sequence in l_∞ which is not in ac, and let T be the matrix whose entries in the first row are the terms of z, and all other entries are zero. For a Banach limit f and $x \in l_\infty$,

$$(T^{**}f)(x) = f(T^*x) = f(\{z_n x_1\}) = f(\{z_n\})x_1,$$

i.e., $T^{**}f = 0 \cdot f + \widehat{f(z)}e_1$. So $T \in \Omega_f$. But $T^*e_1 = z \notin ac$ so $T \notin (l_\infty, ac)$. Since $\phi(f) = f(z)e_1$, this example also shows that the mapping ϕ defined in Theorem(3.21) is, in general, not trivial.

Intersections of subalgebras of Dirac measures turn out to be related to compact operators. The following proposition characterizes compact matrix operators on spaces of interest to us. Some of the items are listed on [26, p.548]. We give them here for ease of reference.

Proposition 3.2 *Let $A = \{a_{nk}\}_{n,k=1}^{\infty}$ be a matrix.*

(i) *If $A \in B(l_1)$, then A is a compact operator if and only if $\sum_{n=1}^{\infty} |a_{nk}|$ converges uniformly in k.*

(ii) *If $A \in B(l_\infty)$, then A is a compact operator if and only if $\sum_{k=1}^{\infty} |a_{nk}|$ converges uniformly in n.*

(iii) *If $A \in B(c_0)$, then A is a compact operator if and only if $\sum_{k=1}^{\infty} |a_{nk}|$ converges uniformly in n.*

(iv) *For any operator $T \in B(c)$, T is compact if and only if $\sum_{k=1}^{\infty} |P_n T e_k|$ converges uniformly in n. In particular, if $A \in B(c)$, then A is compact if and only if $\sum_{k=1}^{\infty} |a_{nk}|$ converges uniformly in n.*

PROOF:

(i). By definition, A is compact if and only if $\overline{A(B_{l_1})}$, the closure of $A(B_{l_1})$, is a norm compact set in l_1. Now $\overline{A(B_{l_1})}$ is the circled closed convex hull of $\{Ae_k\}_{k=1}^{\infty}$. By Mazur' theorem ([26, p.416]), A is compact if and only if $\{Ae_k\}_{k=1}^{\infty}$ is precompact. Because $Ae_k = \{a_{nk}\}_{n=1}^{\infty}$, $\{Ae_k\}_{k=1}^{\infty}$ is precompact if and only if $\sum_{n=1}^{\infty} |a_{nk}|$ converges uniformly in k, this gives (i).

Conclusions (ii) and (iii) follow from (i) and Schauder's theorem ([26, p.485]).

(iv). For $T \in B(c)$, T^* is represented by the matrix:

$$\begin{pmatrix} \chi(T) & \chi_1(T) & \chi_2(T) & \cdots \\ a_1 & P_1Te_1 & P_2Te_1 & \cdots \\ a_2 & P_1Te_2 & P_2Te_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\chi(T) = \lim Te - \sum_k \lim Te_k$, $\chi_n(T) = P_nTe - \sum_k P_nTe_k$ for $n = 1, 2, \ldots$, and $a_k = \lim_n P_nTe_k$ for $k = 1, 2, \ldots$. Schauder's theorem again yields the conclusion. □

Lemma 3.22 *Let $A = \{a_{nk}\} \in B(l_\infty)$ and suppose $\sum_k |a_{nk}|$ converges uniformly in n. Let $p \in \mathbf{N}^*$ and δ_p be the associated Dirac measure on $\beta\mathbf{N}$, regarded as an element of l_∞^*. Then for $x = \{x_k\} \in l_\infty \cong C(\beta\mathbf{N})$,*

$$\delta_p(Ax) = \sum_{k=1}^{\infty} y_k x_k,$$

where $y_k = \delta_p\left(\{a_{nk}\}_{n=1}^{\infty}\right)$, the value of δ_p on the k-th column of the matrix A.

PROOF: Let f and f_m be the continuous extensions to $\beta\mathbf{N}$ of the bounded sequences $\{\sum_{k=1}^{\infty} a_{nk}x_k\}$ and $\{\sum_{k=1}^{m} a_{nk}x_k\}$, $m = 1, 2, 3, \ldots$. Then $f_m \to f$ uniformly on $\beta\mathbf{N}$. Hence,

$$\delta_p(Ax) = \delta_p(f) = \int_{\beta\mathbf{N}} f\, d\delta_p = \lim_{m\to\infty} \int_{\beta\mathbf{N}} f_m\, d\delta_p = \sum_{k=1}^{\infty} x_k y_k.$$ □

Theorem 3.23 *Suppose $T \in B(l_1)$.*

(i) *T is in Ω_p for all $p \in \mathbf{N}^*$ if and only if there is a diagonal matrix $D = diag\{d_n\}$ with $\{d_n\}$ bounded, and a compact operator $T_1 = \{a_{kn}\} \in B(l_1)$ such that $T = D + T_1$.*

(ii) *If T is in Ω_p for all $p \in \mathbf{N}^*$, and $T = D + T_1$ as in* (i), *then $T^{**}\delta_p = \rho_p\delta_p + \widehat{\psi(p)}$, where*

$$\rho_p = \delta_p(\{d_n\}), \quad \text{and } \psi(p) = \{\psi_k(p)\}, \quad \psi_k(p) = \delta_p(\{a_{nk}\}_{n=1}^{\infty}). \tag{3.11}$$

PROOF: Suppose $T = D + T_1$, where $D = \text{diag}\{d_n\}$, $\{d_n\} \in l_\infty$, and $T_1 = \{a_{kn}\}$ is a compact operator in $B(l_1)$. For $p \in \mathbf{N}^*$, define ρ_p and $\psi(p)$ by (3.11). Then for all $x = \{x_k\} \in l_\infty$, we have, by using Lemma(3.22),

$$(T^{**}\delta_p)(x) = \delta_p(T^*x) = \delta_p(D^*x + T_1^*x) = \delta_p\left(\{d_nx_n\} + \left\{\sum_k a_{nk}x_k\right\}\right)$$
$$= \delta_p(\{d_n\})\delta_p(\{x_n\}) + \sum_k \delta_p(\{a_{nk}\}_{n=1}^{\infty})x_k = \rho_p\delta_p(x) + \sum_k \psi_k(p)x_k.$$

So $T^{**}\delta_p = \rho_p\delta_p + \widehat{\psi(p)}$, i.e., $T \in \Omega_p$. This also establishes (ii).

Conversely, Let $T = \{a_{kn}\} \in \bigcap\{\Omega_p \mid p \in \mathbf{N}^*\}$. Since any diagonal matrix in $B(l_1)$ is clearly in $\bigcap\{\Omega_p \mid p \in \mathbf{N}^*\}$, we can, without loss of generality, assume that $a_{kk} = 0$ for all k. Suppose T is not compact, so that $T^* \in B(l_\infty)$ is not compact. Then by Proposition(3.2), $\sum_k |a_{nk}|$ is not uniformly convergent.

There is an increasing sequence $\{n_i''\}$ of positive integers and a constant $\sigma > 0$ such that for all integers m,

$$\liminf_{i\to\infty} \sum_{k=m}^{\infty} |a_{n_i''k}| > \sigma.$$

We can find, by the usual diagonalizing process, a subsequence $\{n_i'\}$ of $\{n_i''\}$ such that $\lim_{i\to\infty} a_{n_i'k} = y_k$ exists for all positive integers k, and $\sum_{k=1}^{\infty} |a_{n_i'k}| > \sigma$ for $n = 1, 2, \ldots$.

Define $k_0 = 0$. Let $n_1 = n_1'$. Find $k_1 \geqslant n_1$, such that

$$\sum_{k=1}^{k_1} |a_{n_1k}| > \sigma, \quad \text{and} \quad \sum_{k=k_1+1}^{\infty} |a_{n_1k}| < \sigma/4.$$

Let N_1 be a positive integer such that, $\inf_{n'_i \geqslant N_1} \sum_{k=k_1+1}^{\infty} |a_{n'_ik}| > \sigma$. Let n_2 be a term in $\{n'_i\}$ so large that it satisfies:

$$n_2 > \max\{k_1, N_1\}, \quad \sum_{k=1}^{k_1} |a_{n_2k} - y_k| < \sigma/4.$$

Choose k_2 such that

$$k_2 \geqslant n_2, \quad \sum_{k=k_1+1}^{k_2} |a_{n_2k}| > \sigma, \text{ and } \sum_{k=k_2+1}^{\infty} |a_{n_2k}| < \sigma/4.$$

Continue this process to obtain sequences $\{n_i\}$ and $\{k_i\}$, where $\{n_i\}$ is a subsequence of $\{n'_i\}$, and for all positive integers i,

$$k_{i-1} < n_i \leqslant k_i,$$

$$\sum_{k=1}^{k_{i-1}} |a_{n_ik} - y_k| < \sigma/4,$$

$$\sum_{k=k_{i-1}+1}^{k_i} |a_{n_ik}| > \sigma,$$

$$\sum_{k=k_i+1}^{\infty} |a_{n_ik}| < \sigma/4. \tag{3.12}$$

We now show that if $p \in \overline{\{n_i\}}$, then $T \notin \Omega_p$. Suppose, to the contrary, that

$$T^{***}\delta_p = \rho_p\delta_p + \hat{z} \text{ for some } z \in l_1. \tag{3.13}$$

Evaluating (3.13) at e_k shows that $z_k = \delta_p(\{a_{nk}\}_{n=1}^{\infty}) = y_k$.

Next, take $x = \{x_k\} \in l_\infty$ with x_k defined by $x_k = \operatorname{sgn}(a_{n_ik})$ for $k_{i-1} < k \leqslant k_i$ (we use the convention $\operatorname{sgn}(0) = 0$). Since we have $k_{i-1} < n_i \leqslant k_i$ for all i and $a_{kk} = 0$ for all k, we have $x_{n_i} = 0$ for all i. Thus, $\delta_p(x) = 0$.

There is a positive integer K such that $\sum_{k=K+1}^{\infty} |y_k| < \sigma/4$. Now for i large enough that $k_{i-1} > K$, we have, using (3.12),

$$\left|\sum_{k=1}^{\infty} a_{n_i k} x_k - \sum_{k=1}^{\infty} y_k x_k\right|$$

$$\geqslant \left|\sum_{k=k_{i-1}+1}^{k_i} a_{n_i k} x_k\right| - \left|\sum_{k=1}^{k_{i-1}} a_{n_i k} x_k - \sum_{k=1}^{k_{i-1}} y_k x_k\right| - \left|\sum_{k=k_i+1}^{\infty} a_{n_i k} x_k\right| - \left|\sum_{k=k_{i-1}+1}^{\infty} y_k x_k\right|$$

$$\geqslant \sum_{k=k_{i-1}+1}^{k_i} |a_{n_i k}| - \sum_{k=1}^{k_{i-1}} |a_{n_i k} - y_k| - \sum_{k=k_i+1}^{\infty} |a_{n_i k}| - \sum_{k=K+1}^{\infty} |y_k|$$

$$\geqslant \sigma - \sigma/4 - \sigma/4 - \sigma/4 = \sigma/4.$$

However, since $(T^{**}\delta_p)(x)$ is a subsequential limit of $\{\sum_{k=1}^{\infty} a_{n_i k} x_k\}_i$ and $\delta_p(x) = 0$, we have $(T^{**}\delta_p)(x) \neq \rho_p \delta_p(x) + \hat{y}(x)$. So the theorem is proved. □

Corollary 3.24 *An operator $T \in B(l_1)$ is in Γ_p for all $p \in \mathbf{N}^*$ if and only if there is a diagonal matrix D and a matrix T_0, whose transpose maps l_∞ into c_0, such that $T = D + T_0$.*

PROOF: Now, $T \in \Gamma_p$ if and only if $T \in \Omega_p$ and $\psi(p) = 0$ where $\psi(p)$ is defined in (3.11). So T is in $\bigcap\{\Gamma_p \mid p \in \mathbf{N}^*\}$ if and only if both of the following two conditions are true: (a) $T \in \bigcap\{\Omega_p \mid p \in \mathbf{N}^*\}$, and (b) $\psi(p) = 0$ for all $p \in \mathbf{N}^*$. By Theorem(3.23), we can write $T = D + T_0 = \text{diag}\,\{d_n\} + \{a_{kn}\}$, where $\{d_n\} \in l_\infty$ and $A = \{a_{kn}\} \in B(l_1)$ is compact. Also by Theorem(3.23), condition (b) means that $\psi_k(p) = 0$ for all p and all k. This is equivalent to $\{a_{nk}\}_n \in c_0$ for all k. It follows from Schur's theorem that the transpose of T_0 maps l_∞ into c_0. □

Chapter 4

Bounded Linear Operators on Sequences with Terms in a Banach Space

4.1 Introduction

For any Banach space X, let $s(X)$ be the set of all sequences $\mathbf{x} = \{x_k\}$ with $x_k \in X$; $k = 1, 2, 3, \ldots$. For $\mathbf{x} \in s(X)$ define

$$\|\mathbf{x}\|_\infty = \sup_{k \geq 1} \|x_k\|$$

and, for $1 \leq p < \infty$,

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{\infty} \|x_k\|^p \right)^{1/p}.$$

In calculations involving $\|\cdot\|_p$, we are often able to combine the cases $1 \leq p < \infty$ and $p = \infty$ by interpreting expressions like $(\sum_{k=i}^{\infty} \|x_k\|^p)^{1/p}$ as $\sup_{k \geq 1} \|x_k\|$ when $p = \infty$. We shall do this without further comment.

We shall be concerned particularly with the following subspaces of $s(X)$: $l_\infty(X)$, the bounded sequences in X, that is those sequences $\mathbf{x}$ for which $\|\mathbf{x}\|_\infty < \infty$, $c(X)$ consisting of the convergent sequences in X, $c_0(X)$ consisting of the null sequences in X, and $l_p(X)$, $1 \leq p < \infty$, consisting of those sequences $\mathbf{x}$ for which $\|\mathbf{x}\|_p < \infty$.

The spaces $l_\infty(X)$, $c(X)$ and $c_0(X)$ are Banach spaces under $\|\cdot\|_\infty$ and $l_p(X)$, for $1 \le p < \infty$, is a Banach space under $\|\cdot\|_p$.

For a Banach space X and $x \in X$, define

$$\mathbf{e}_k(x) = \{0, 0 \ldots 0, x, 0, 0, \ldots\} \in s(X),$$

w·ere x is in the k-th position, and

$$\mathbf{e}(x) = \{x, x, x, \ldots\} \in s(X).$$

On the other hand, for $\mathbf{x} = \{x_k\} \in s(X)$, define $P_n(\mathbf{x}) = x_n \in X$.

In [36] Maddox discusses infinite matrices of operators between various sequence spaces whose elements have their terms in Banach spaces. More precisely, [36, page 8f], he considers the following situation.

> Let X and Y be Banach spaces and $A = \{A_{nk}\}$ be an infinite matrix of linear, but not necessarily bounded, operators A_{nk} from X to Y. Suppose E is a nonempty subset of $s(X)$ and F is a nonempty subset of $s(Y)$. We define the matrix class (E, F) by saying that $A \in (E, F)$ if and only if, for every $\mathbf{x} = \{x_k\} \in E$,
>
> $$A_n\mathbf{x} = \sum_{k=1}^{\infty} A_{nk}x_k$$
>
> converges in the norm topology of Y, for each n, and the sequence $A\mathbf{x} = \{\sum A_{nk}x_k\}_{n\in\mathbb{N}}$ belongs to F.

In this chapter we restrict our attention to the case where the operators A_{nk} are bounded. One reason for doing this is that even if we do not require boundedness from the beginning, the fact that $A = \{A_{nk}\} \in (E, F)$ for all common pairs of E, F will imply that A_{nk} must be bounded for k's that are large enough (see, for example, theorems (4.2), (4.6), (4.8), (4.9), (4.10) of [36]. Another reason for doing it is that we will define a bounded linear operator and can possibly find the norm of A by requiring all A_{nk}'s to be bounded. We also consider some bounded linear operators between E and F which are not given by matrices in the manner described above. Moreover, in considering the series $\sum_{k=1}^{\infty} A_{nk}x_k$, a number of possibilities arise concerning the type of convergence one might desire. The series could, for example, be required to converge weakly, in norm, unconditionally in norm, or absolutely. Asking for absolute convergence seems to ask too much while having unconditional convergence in norm seems most satisfactory. Indeed, we show that in many cases of interest, for instance, when E has the Orlicz-Pettis property (see Definition(1.2)), weak convergence leads automatically to unconditional convergence in norm. If E fails to have the Orlicz-Pettis property, for example, in the case of principal interest in this chapter, $E = c(X)$, the situation is more complicated but can be managed by requiring that Y contain no copy of c_0, which includes the case where Y is weakly sequentially complete. We also show by example that even when $Y = l_1$, it is possible that the different requirements for convergence are not equivalent.

4.2 Some Preliminary Results

Lemma 4.1 *Let X and Y be Banach spaces and $\Lambda : X \longrightarrow Y$ be a linear map which is continuous when X has the norm topology and Y has the weak topology. Then Λ is continuous when Y has the norm topology. (Briefly: If Λ is strong-weak continuous, then it is strong-strong continuous.)*

PROOF: From [41, Theorem 1.32] we see that Λ is bounded. Thus $\Lambda(B_X)$ is weakly bounded. Now from the Corollary on [41, Corollary page 69] we find that
$\sup_{\|x\|\leq 1} \|\Lambda x\| < \infty$. □

We will use the notation $\lim^\sigma$ to indicate weak convergence of a sequence, and $\sum^\sigma$ to indicate weak convergence of a series.

Lemma 4.2 *Let X and Y be Banach spaces and let E denote any of the spaces $c_0(X)$, $c(X)$, or $l_p(X)$ $1 \leq p \leq \infty$. Suppose $\Lambda_k \in B(X,Y)$ and that $\sum_{k=1}^\infty \Lambda_k x_k$ converges weakly for every $\mathbf{x} = \{x_k\} \in E$. Then $\Lambda\mathbf{x} = \sum_{k=1}^\infty \Lambda_k x_k$ is a strong-strong continuous operator from E to Y.*

PROOF: Consider

$$L_n\mathbf{x} = \sum_{k=1}^n \Lambda_k x_k \quad \text{for } \mathbf{x} = \{x_k\} \in E.$$

We see that $L_n : E \longrightarrow Y$ is strong-strong continuous because

$$\|L_n\mathbf{x}\| \leq \sum_{k=1}^n \|\Lambda_k\| \, \|x_k\| \leq (\sum_{k=1}^n \|\Lambda_k\|^{p'})^{1/p'} \, \|\mathbf{x}\|_p,$$

and so L_n is strong-weak continuous. Now, $\Lambda\mathbf{x} = \lim^\sigma_{n\to\infty} L_n\mathbf{x}$, so [41, Theorem 2.8] shows that Λ is strong-weak continuous. We then use Lemma(4.1) to conclude that Λ is strong-strong continuous. □

Definition 4.1 *Let $\sum_n x_n$ be a formal series in a Banach space X. Then $\sum_n x_n$ is said to be* **unconditionally convergent in norm** *if $\sum_n x_{\pi(n)}$ is convergent in norm for each permutation π of the natural numbers. The series $\sum_n x_n$ is said to be* **weakly unconditionally Cauchy**, *sometimes abbreviated as 'WUC', if, for each $x^* \in X^*$, $\sum_n |x^* x_n| < \infty$.*

The following two theorems, one due to Bessaga and Pelczyński [3, Theorem 5] and the other due to Orlicz and Pettis [38, Theorem 2.32], are pivotal to the study in this and the next chapter. We state them here for easy reference. One can also see [24] for their proofs.

Theorem 4.3 *A Banach space X has a weakly unconditionally Cauchy series that is not unconditionally convergent if and only if X contains a subspace which is isomorphic to c_0.*

Theorem 4.4 *Let $\sum_n x_n$ be a series whose terms belong to a Banach space X. Suppose for each increasing sequence $\{n_k\}$ of integers, $\sum_k x_{n_k}$ is weakly convergent. Then $\sum_n x_n$ is unconditionally convergent in norm.*

Some Banach spaces have a property that makes it easier to apply the Orlicz-Pettis theorem:

Definition 4.2 *Let X be a Banach space and $E \subset s(X)$. Then E is said to have the* **Orlicz-Pettis property** *if, for every $\mathbf{x} = \{x_k\} \in E$ and every increasing sequence of natural numbers $\{k_i\}$, the sequence $\mathbf{y} = \{y_k\} \in E$ where $y_{k_i} = x_{k_i}$ and $y_k = 0$ otherwise.*

The spaces $c_0(X)$ and $l_p(X)$ for $1 \leq p \leq \infty$ all have the Orlicz-Pettis property, while the space $c(X)$ does not.

Lemma 4.5 *Let X and Y be Banach spaces and let $E \subset s(X)$ have the Orlicz-Pettis property. Suppose $\Lambda_k \in B(X,Y)$ $k = 1,2,\ldots$ and $\sum_{k=1}^{\infty} \Lambda_k x_k$ converges weakly for every $\mathbf{x} = \{x_k\} \in E$. Then $\sum_{k=1}^{\infty} \Lambda_k x_k$ converges unconditionally in norm for every $\mathbf{x} = \{x_k\} \in E$.*

PROOF: Since E has the Orlicz-Pettis property, it follows that $\sum \Lambda_{k_i} x_{k_i}$ converges weakly for every increasing sequence $\{k_i\}$ of natural numbers. Theorem(4.4) now yields the conclusion. $\square$

That even under the hypotheses of Lemma(4.5) absolute convergence may fail is seen by taking $X = \mathbf{R}$, the real numbers, $Y = c_0$ and $\Lambda_k x = x e_k$. Then for $\mathbf{x} = \{x_k\} \in$

$c_0(\mathbf{R}) = c_0$ we have $\sum_{k=1}^{\infty} \Lambda_k x_k = \sum_{k=1}^{\infty} x_k e_k$ which is unconditionally convergent ($\{e_k\}$ is an unconditional basis for c_0, see, for example, [42, p.396]) while with $x_k = 1/k$ the convergence fails to be absolute.

Lemma 4.6 *Let X and Y be Banach spaces and suppose Y contains no copy of c_0. Suppose $\Lambda_k \in B(X,Y)$ $k = 1,2,\ldots$ and $\sum_{k=1}^{\infty} \Lambda_k x_k$ converges weakly for every $\mathbf{x} = \{x_k\} \in c(X)$. Then $\sum_{k=1}^{\infty} \Lambda_k x_k$ converges unconditionally in norm for every $\mathbf{x} = \{x_k\} \in c(X)$*

PROOF: For $f \in Y^*$ we have, by Lemma(4.2), that $f\Lambda \in c^*(X)$, where $\Lambda\mathbf{x} = \sum^{\infty} \Lambda_k x_k$ for $\mathbf{x} = \{x_k\} \in c(X)$. Now, for $z = \{z_k\} \in c_0$, we have $\sum_{k=1}^{\infty} z_k f \Lambda_k x_k = \sum_{k=1}^{\infty} f \Lambda_k z_k x_k$ converges. Hence $\sum_{k=1}^{\infty} |f \Lambda_k x_k| < \infty$. So $\sum_{k=1}^{\infty} \Lambda_k x_k$ is weakly unconditionally Cauchy. Theorem(4.3) yields the desired conclusion. □

If Y contains a copy of c_0, then the conclusion of Lemma(4.6) can fail as the following example shows. Take $X = \mathbb{R}$ and $Y = c_0$. Take, for $x \in \mathbb{R}$, $\Lambda_1 x = -x e_1$ and $\Lambda_k x = x e_{k-1} - x e_k$ for $k > 1$. Then for $\mathbf{x} = \{x_k\} \in c(\mathbf{R}) = c$, we have $\sum_{k=1}^{n} \Lambda_k x_k = \{x_2 - x_1, x_3 - x_2, \ldots x_n - x_{n-1}, -x_n, 0, 0, \ldots\}$ which converges weakly in c_0 to $\{x_{k+1} - x_k\} \in c_0$. (In fact $\sum_{k=1}^{\infty} \Lambda_k x_k$ is also weakly unconditionally Cauchy for $\mathbf{x} = \{x_k\} \in c$). For the sequence $e = \{1, 1, 1, \ldots\}$, however, we have $\sum_{k=1}^{n} \Lambda_k x_k = -e_n$ so that $\sum_{k=1}^{\infty} \Lambda_k x_k$ fails to converge in norm in c_0. In fact, $\sum_{k=1}^{\infty} \Lambda_k x_k$ fails to converge in norm in c_0 whenever $\mathbf{x} = \{x_k\} \notin c_0$.

Lemma 4.7 *Let X and Y be Banach spaces and $\Lambda_k \in B(X,Y)$. Suppose $\sum_{k=1}^{\infty} \Lambda_k x_k$ converges in norm in Y for every $\mathbf{x} = \{x_k\} \in c_0(X)$. Then $\{\sum_{k=1}^{\infty} \Lambda_k x_k\}$ is weakly unconditionally Cauchy for every $\mathbf{x} = \{x_k\} \in l_\infty(X)$, and hence unconditionally convergent in norm for every $\mathbf{x} = \{x_k\} \in l_\infty(X)$ if, in addition, Y contains no copy of c_0.*

PROOF: Let $f \in Y^*$ and $\{z_k\} \in c_0$, then for $\{x_k\} \in l_\infty(X)$ we have

$$\sum_{k=1}^{\infty} z_k f \Lambda_k x_k = \sum_{k=1}^{\infty} f \Lambda_k z_k x_k,$$

noting that the latter series converges because $\{z_k x_k\} \in c_0(X)$. Hence $\sum_{k=1}^{\infty} |f\Lambda_k x_k| < \infty$. The statement about unconditional convergence follows from Theorem(4.3). □

If $Y = c_0$, however, we can take $X = \mathbb{R}$, and $\Lambda_k x = x e_k$ for $x \in \mathbb{R}$. then for $\mathbf{x} = \{x_k\} \in c_0$ we have $\mathbf{x} = \sum_{k=1}^{\infty} \Lambda_k x_k$ where the series converges unconditionally in the norm of c_0. But for $\mathbf{x} = \{x_k\} \in l_\infty$, $\{\sum_{k=1}^{\infty} \Lambda_k x_k\}$ does not have a weak limit point in c_0 when $\mathbf{x} \notin c_0$.

To study a sequence of operators as an operator on $s(X)$, the concept of the group norm was introduced in [39], see also [36, page 5].

Definition 4.3 *Let X and Y be Banach spaces and $\mathbf{T} = \{T_k\}$ be a sequence of operators in $B(X,Y)$. The* **group norm** *of $\mathbf{T}$ is defined to be*

$$\|\mathbf{T}\|_{GRP} = \|\{T_k\}\| = \sup \|\sum_{k=1}^{n} T_k x_k\|,$$

where the supremum is taken over $n = 1, 2, \ldots$ and all finite sequences $\{x_k\}_{k=1}^{n}$ of elements in B_X.

We use $GRP(X,Y)$ to denote the set of all sequences $\{T_k\} \subset B(X,Y)$ for which $\|\{T_k\}\| < \infty$. Maddox observed in [36] that $GRP(X,Y)$ is a Banach space under the natural linear structure and the norm $\|\cdot\|_{GRP}$.

Here is an immediate property of the group norr

Proposition 4.1 *Let $\{T_k\}$ be a sequence of bounded linear operators from a Banach space X to another Banach space Y, then*

$$\|\{T_k\}\| = \sup_{y^* \in B_{Y^*}} \sum_{k=1}^{\infty} \|T_k^* y^*\|, \quad \textit{and} \quad \|\{T_k^*\}\| = \sup_{x \in B_X} \sum_{k=1}^{\infty} \|T_k x\|.$$

PROOF:

$$\sup_{y^*\in B_{Y^*}} \sum_{k=1}^{\infty} \|T_k^* y^*\| = \sup_{y^*\in B_{Y^*}} \sup_{n\geqslant 1} \sum_{k=1}^{n} \|T_k^* y^*\| = \sup_{y^*\in B_{Y^*}} \sup_{n\geqslant 1} \sup_{x_k\in B_X} \sum_{k=1}^{n} (T_k^* y^*)(x_k)$$

$$= \sup_{n\geqslant 1} \sup_{x_k\in B_X} \sup_{y^*\in B_{Y^*}} \sum_{k=1}^{n} y^*(T_k x_k) = \sup_{n\geqslant 1} \sup_{x_k\in B_X} \left\| \sum_{k=1}^{n} T_k x_k \right\| = \|\{T_k\}\|.$$

The proof of the second part is similar. □

4.3 Some Representation Theorems

The principal aim of this section is to obtain a representation theorem for operators in $B(c(X), c(Y))$ which is analogous to that obtained by Crawford in [22] for operators in $B(c)$. See also Taylor [44, Theorem 4, 51-D]. From the above discussion we shall from time to time require Y to contain no copy of c_0.

We will use lim to denote the bounded linear operator $c(X) \to X$: $\lim \mathbf{x} = lim_k x_k$ for $\mathbf{x} = \{x_k\} \in c(X)$.

We first prove two lemmas concerning group norms.

Lemma 4.8 *Let X and Y be Banach spaces and $\{T_k\}$ be a sequence of operators in $B(X, Y)$. Then*

$$\|\{T_k\}\| = \sup_{f\in B_{Y^*}} \|\{fT_k\}\|.$$

PROOF: Let $f \in B_{Y^*}$ and $x_1, \ldots, x_n \in B_X$. Then

$$\|\sum_{k=1}^{n} fT_k x_k\| \leq \|f\| \, \|\sum_{k=1}^{n} T_k x_k\| \leq \|\{T_k\}\|.$$

Hence,

$$\sup_{f\in B_{Y^*}} \|\{fT_k\}\| \leq \|\{T_k\}\|.$$

Let $x_k \in B_X$ for $k = 1, 2, \ldots, n$ and choose, using the Hahn-Banach Theorem, $f \in Y^*$ with $\|f\| = 1$ and $f(\sum_{k=1}^n T_k x_k) = \|\sum_{k=1}^n T_k x_k\|$. It now follows that

$$\sup_{f \in B_{Y^*}} \|\{fT_k\}\| \geq \|\{T_k\}\| \qquad \square$$

Lemma 4.9 *Let X be a Banach space and $\{f_k\}$ be a sequence in X^*. Then*

$$\|\{f_k\}\| = \sum_{k=1}^{\infty} \|f_k\|.$$

PROOF: Let n be arbitrary and choose $x_k \in B_X$, $k = 1, \ldots, n$. Then

$$|\sum_{k=1}^{n} f_k x_k| \leq \sum_{k=1}^{\infty} \|f_k\|.$$

Hence,

$$\|\{f_k\}\| \leq \sum_{k=1}^{\infty} \|f_k\|.$$

Conversely, for $\epsilon > 0$, choose $x_k \in S_X$ such that $|f_k x_k| > \|f_k\| - \epsilon/2^k$. Let $z_k = \arg(f_k(x_k))x_k$. Then

$$|\sum_{k=1}^{\infty} f_k z_k| = \sum_{k=1}^{\infty} |f_k x_k| \geq \sum_{k=1}^{\infty} \|f_k\| - \epsilon.$$

Thus,

$$\|\{f_k\}\| \geq \textstyle\sum_{k=1}^{\infty} \|f_k\|. \qquad \square$$

Lemma 4.10 *Let X and Y be Banach spaces, and $f : c_0(X) \longrightarrow Y$ be a bounded linear operator. Then for $\mathbf{x} = \{x_k\} \in c_0(X)$,*

$$f(\mathbf{x}) = \sum_{k=1}^{\infty} f\mathbf{e}_k(x_k), \tag{4.1}$$

where the series converges unconditionally in norm in Y, and $\|f\| = \|\{f\mathbf{e}_k\}\|$.

PROOF: The unconditional convergence of the series in equation (4.1) follows easily from Lemma(4.5). For any n choose $x_k \in B_X$, $k = 1, \ldots, n$. If $x = \sum_{k=1}^n \mathbf{e}_k(x_k)$, then $\|\mathbf{x}\| \leq 1$ and

$$\|\sum_{k=1}^{n} f\mathbf{e}_k(x_k)\| = \|f(\mathbf{x})\| \leq \|f\|,$$

so that $\|\{f\mathbf{e}_k\}\| \leq \|f\|$. Conversely, given $\epsilon > 0$, choose $\mathbf{x} = \{x_k\}$ with $\|\mathbf{x}\| = 1$ and

$$\|f\| - \epsilon < \|f(\mathbf{x})\| = \|\sum_{k=1}^{\infty} f\mathbf{e}_k(x_k)\| \leq \|\{f\mathbf{e}_k\}\|$$

Thus, $\|f\| \leq \|\{f\mathbf{e}_k\}\|$. □

Lemma 4.11 *Let X and Y be Banach spaces and suppose Y contains no copy of c_0. Let $f \in B(c(X), Y)$. Define,*

$$\psi(f)x = f\mathbf{e}(x) - \sum_{k=1}^{\infty} f\mathbf{e}_k(x)$$

for $x \in X$. Then $\psi(f) \in B(X, Y)$.

PROOF: The linearity of $\psi(f)$ is clear. Now $f|_{c_0(X)}$ is a bounded linear operator from $c_0(X)$ to Y so that, for $\mathbf{x} = \{x_k\} \in c_0(X)$, $\sum_{k=1}^{\infty} f\mathbf{e}_k(x_k)$ is convergent in norm in Y. Lemma(4.7) shows that $\sum_{k=1}^{\infty} f\mathbf{e}_k(x_k)$ is unconditionally convergent in norm in Y for each $\mathbf{x} = \{x_k\} \in c(X)$. Hence $\{f\mathbf{e}_k\} \in c^{\upsilon}(X) = c^{\beta}(X)$. (See Definition(5.1).) From [36, page 29] we see that $\|\{f\mathbf{e}_k\}\| < \infty$. Hence, because

$$\|\psi(f)x\| \leq (\|f\mathbf{e}\| + \|\{f\mathbf{e}_k\}\|)\|x\|,$$

we see that $\psi(f)$ is bounded. □

Lemma 4.12 *Let X and Y be Banach spaces and suppose Y contains no copy of c_0. Suppose $f \in B(c(X), Y)$. Then for $\mathbf{x} = \{x_k\} \in c(X)$,*

$$f(\mathbf{x}) = \psi(f)(\lim \mathbf{x}) + \sum_{k=1}^{\infty} f\mathbf{e}_k(x_k) \tag{4.2}$$

where the series converges unconditionally in norm in Y. Moreover,

$$\|f\| = \|\{\psi(f), f\mathbf{e}_1, f\mathbf{e}_2, \ldots\}\|.$$

PROOF: If $\mathbf{x} = \{x_k\} \in c(X)$, then

$$\mathbf{x} = \mathbf{e}(\lim \mathbf{x}) + \sum_{k=1}^{\infty} \mathbf{e}_k(x_k - \lim \mathbf{x}).$$

We saw in the proof of Lemma(4.11) that $\sum_{k=1}^{\infty} f\mathbf{e}_k(x_k)$ converges unconditionally in norm in Y for all $\mathbf{x} = \{x_k\} \in c(X)$. Hence (4.2) follows.

Let $\epsilon > 0$ and choose $\mathbf{x} = \{x_k\} \in c(X)$ with $\|\mathbf{x}\| = 1$ and $\|f(\mathbf{x})\| > \|f\| - \epsilon$, then

$$\|f\| - \epsilon < \|f(\mathbf{x})\| = \left\|\psi(f)(\lim \mathbf{x}) + \sum_{k=1}^{\infty} f\mathbf{e}_k(x_k)\right\|.$$

It follows that $\|f\| \leq \|\{\psi(f), f\mathbf{e}_1, f\mathbf{e}_2, \ldots\}\|$. Conversely, for any fixed $\epsilon > 0$ and $x, x_1, x_2, \ldots, x_n \in B_X$ such that

$$\|\psi(f)(x) + \sum_{k=1}^{n} f\mathbf{e}_k(x_k)\| \geqslant \|\{\psi(f), f\mathbf{e}_1, f\mathbf{e}_2, \ldots\}\| - \epsilon/2,$$

pick $n_1 \in \mathbf{N}$, $n_1 > n$ such that $\|\sum_{k=n_1}^{\infty} f\mathbf{e}_k(x)\| \leqslant \epsilon/2$, then

$$\|\{\psi(f), f\mathbf{e}_1, f\mathbf{e}_2, \ldots\}\| \leqslant \left\|\psi(f)(x) + \sum_{k=1}^{n} f\mathbf{e}_k(x_k)\right\| + \epsilon/2$$

$$\leqslant \left\|\psi(f)(x) + \sum_{k=1}^{n} f\mathbf{e}_k(x_k) + \sum_{k=n_1}^{\infty} f\mathbf{e}_k(x)\right\| + \epsilon = \|f(\mathbf{x})\| + \epsilon,$$

where $\mathbf{x} = \{x'_k\}$ is the sequence in $c(X)$ with

$$x'_k = \begin{cases} x_k & \text{if } 1 \leqslant k \leqslant n, \\ 0 & \text{if } n < k < n_1, \\ x & \text{if } k \geqslant n_1. \end{cases}$$

Thus $\|\{\psi(f), f\mathbf{e}_1, f\mathbf{e}_2, \ldots\}\| \leqslant \|f\|$. □

Theorem 4.13 *Let X and Y be Banach spaces. Suppose $T \in B(c_0(X), l_\infty(Y))$. Then T is given by the matrix $\{A_{nk}\}$, where $A_{nk} = P_nT\mathbf{e}_k \in B(X, Y)$, in the sense that, for $\mathbf{x} = \{x_k\} \in c_0(X)$,*

$$T\mathbf{x} = \left\{\sum_{k=1}^{\infty} A_{nk}x_k\right\} \tag{4.3}$$

with the series converging unconditionally in norm in Y. Moreover,

$$\|T\| = \sup_{n\geq 1} \|\{A_{nk}\}\| \tag{4.4}$$

Conversely, if $A_{nk} \in B(X,Y)$ for $n, k = 1, 2, \dots$ and $\sup_{n\geq 1} \|\{A_{nk}\}\| < \infty$, then the operator T defined by (4.3) is in $B(c_0(X), l_\infty(Y))$.

PROOF:. Suppose that $T \in B(c_0(X), l_\infty(Y))$. Equation (4.3) and the unconditional convergence of the series follow from Lemma(4.10) Now $T\mathbf{x} = \{P_n T\mathbf{x}\}$ for $\mathbf{x} \in c_0(X)$. Thus $\sup_{n\geq 1} \|P_n T\mathbf{x}\| < \infty$ for each $\mathbf{x} \in c_0(X)$ so, by the Banach-Steinhaus theorem [41, Theorem 2.6] $H = \sup_{n\geq 1} \|P_n T\| < \infty$. Lemma(4.10) shows that $\|P_n T\| = \|\{A_{nk}\}\|$. Hence

$$\|T\| \leq H = \sup_{n\geq 1} \|\{A_{nk}\}\| < \infty.$$

On the other hand, let $\epsilon > 0$ and choose n_0 so that $\|\{A_{n_0 k}\}\| > H - \epsilon$; then choose $x_1, \dots, x_m \in S_X$ such that $\|\sum_{k=1}^m A_{n_0 k} x_k\| > \|\{A_{n_0 k}\}\| - \epsilon$. Now with $\mathbf{x} = \{x_1, \dots, x_m, 0, 0, \dots\}$ we have $\|\mathbf{x}\| \leq 1$ and $\|T\mathbf{x}\| \geq \|\sum_{k=1}^m A_{n_0 k} x_k\| > H - 2\epsilon$. Thus, $\|T\| \geq H$ and (4.4) holds.

Finally, suppose $A_{nk} \in B(X,Y)$ and $\sup_{n\geq 1} \|\{A_{nk}\}\| < \infty$. Let $\mathbf{x} = \{x_k\} \in c_0(X)$. Then from [36, Proposition 2.3(c)],

$$\left\|\sum_{k=1}^{\infty} A_{nk} x_k\right\| \leq \|\{A_{nk}\}\| \|\mathbf{x}\| \leq \sup_{n\geq 1} \|\{A_{nk}\}\| \|\mathbf{x}\| < \infty.$$

Hence, $T\mathbf{x} = \{\sum_{k=1}^\infty A_{nk} x_k\} \in l_\infty(Y)$ and $T \in B(c_0(X), l_\infty(Y))$. □

Combining (4.4) with Lemma(4.10) we have the following corollary.

Corollary 4.14 *Let X, Y and T be as in Theorem(4.13). Then*

$$\|\{T\mathbf{e}_k\}\| = \sup_{n\geq 1} \|\{A_{nk}\}\|.$$

Corollary 4.15 *Let X, Y and T be as in Theorem(4.13), but with Y containing no copy of c_0. Then T may be regarded as an operator in $B(l_\infty(X), l_\infty(Y))$.*

PROOF: We use Lemma(4.7) to see that $T\mathbf{x}$ exists for all $\mathbf{x} \in l_\infty(X)$, and then observe that

$$\|T\mathbf{x}\|_\infty = \sup_{n\geq 1}\left\|\sum_{k=1}^{\infty} A_{nk}x_k\right\| \leq \sup_{n\geq 1} \|\{A_{nk}\}\| \, \|\mathbf{x}\|_\infty. \qquad \square$$

The conclusion of Corollary(4.15) can fail if Y contains a copy of c_0. For example, if we take $X = \mathbf{R}$, $Y = c_0$, $A_{nk} = 0$ for $k < n$ and $A_{nk} = \mathbf{e}_k$ for $k \geq n$. Then $T = \{A_{nk}\}$ defines a bounded linear operator from c_0 to $c_0(c_0) \subset l_\infty(c_0)$; but $\sum_{k=n}^{\infty} \mathbf{e}_k(x_k)$ does not converge in norm in c_0 unless $\mathbf{x} \in c_0$.

Theorem 4.16 *Let X and Y be Banach spaces and $T \in B(c_0(X), c(Y))$. Then, for $\mathbf{x} = \{x_k\} \in c_0(X)$,*

$$T\mathbf{x} = \left\{\sum_{k=1}^{\infty} A_{nk}x_k\right\} \tag{4.5}$$

where $A_{nk} = P_nT\mathbf{e}_k$ and the series converge unconditionally in norm in Y. Also,

$$\|T\| = \sup_{n\geq 1} \|\{A_{nk}\}\| \tag{4.6}$$

and

$$\lim_{n\to\infty} A_{nk}(x) \text{ exists} \tag{4.7}$$

for $k = 1, 2, \ldots$ and $x \in X$.

Moreover, if we define $A_k(x) = \lim_{n\to\infty} A_{nk}(x)$ for $k = 1, 2, \ldots$ and $x \in X$, then $A_k \in B(X, Y)$, $\|\{A_k\}\| \leq \|T\|$, $A\mathbf{x} = \sum_{k=1}^{\infty} A_k x_k$ converges unconditionally in norm in Y for $\mathbf{x} = \{x_k\} \in c_0(X)$, and $\lim T\mathbf{x} = A\mathbf{x}$ for $\mathbf{x} = \{x_k\} \in c_0(X)$. Further, $A \in B(c_0(X), Y)$ and $\|A\| = \|\{A_k\}\|$.

Conversely, if $A_{nk} \in B(X,Y)$ for $n,k = 1,2,\ldots$, $\sup_{n\geq 1} \|\{A_{nk}\}\| < \infty$ and (4.7) holds for $k = 1,2,\ldots$ and $x \in X$, then the operator T defined by (4.5) is in $B(c_0(X), c(Y))$.

PROOF: Suppose $T \in B(c_0(X), c(Y))$. That (4.6) holds follows from Theorem(4.13). For $x \in X$ we have $\mathbf{e}_k(x) \in c_0(X)$ so $T\mathbf{e}_k(x) = \{A_{nk}(x)\} \in c(Y)$ which establishes (4.7). Now

$$\|A_k(x)\| \leq \sup_{n\geq 1} \|A_{nk}(x)\| \leq \sup_{n\geq 1} \|A_{nk}\|\|x\| \leq \|T\|\|x\|.$$

Hence $\sup_{k\geq 1} \|A_k\| < \infty$, so, *a fortiori*, $\|A_k\| < \infty$. Suppose $x_1, x_2, \ldots, x_m \in B_X$, then

$$\left\|\sum_{k=1}^{m} A_k x_k\right\| = \lim_{n\to\infty} \left\|\sum_{k=1}^{m} A_{nk} x_k\right\| \leq \|T\|.$$

Thus $\|\{A_k\}\| \leq \|T\| < \infty$ so that $\sum_{k=1}^{\infty} A_k x_k$ is strongly convergent by [36, page19], and hence, by Lemma(4.5), unconditionally convergent in norm in Y, for all $\mathbf{x} = \{x_k\} \in c_0(X)$ and, by Lemma(4.2) $A \in B(c_0(X), Y)$. That $\|A\| = \|\{A_k\}\|$ follows from Lemma(4.10).

To show that $\lim T\mathbf{x} = A\mathbf{x}$, let $\epsilon > 0$ and choose K so that $\|x_k\| < \epsilon/(2\|T\|)$ for $k > K$. Now

$$\sum_{k=1}^{\infty} A_{nk} x_k - \sum_{k=1}^{\infty} A_k x_k = \sum_{k=1}^{\infty} (A_{nk} - A_k) x_k$$

$$= \sum_{k=1}^{K} (A_{nk} - A_k) x_k + \sum_{k=K+1}^{\infty} (A_{nk} - A_k) x_k = U + V \text{ say.}$$

We have, by Proposition 2.3 of [36],

$$\|V\| \leq \|\{A_{nk} - A_k\}\| \frac{\epsilon}{2\|T\|} < \epsilon$$

and for K fixed $U \to 0$ as $n \to \infty$ and the result follows.

Conversely, if $A_{nk} \in B(X,Y)$ and $\sup_{n\geq 1} \|\{A_{nk}\}\| < \infty$, then the operator T defined by (4.5) is, first of all, in $B(c_0(X), l_\infty(Y))$ by Theorem(4.13). With (4.7) also holding we see from the above proof that in fact $T \in B(c_0(X), c(Y))$. $\square$

Remark If, in Theorem(4.16) $c(Y)$ is replaced by $c_0(Y)$, then $A_k(x) = 0$ for $k = 1, 2, \ldots$ and $x \in X$.

Suppose that X and Y are Banach Spaces and Y contains no copy of c_0. For $T \in B(c(X), c(Y))$ and $x \in X$, let $T_{nk}(x) = P_n T\mathbf{e}_k(x)$ and $T_k(x) = \lim_{n\to\infty} T_{nk}(x)$. Lemma(4.7) shows that $\sum_{k=1}^{\infty} T_{nk}(x_k)$ and, Theorem(4.16) that, $\sum_{k=1}^{\infty} T_k(x_k)$, converge unconditionally in norm in Y for all $\mathbf{x} = \{x_k\} \in l_\infty(X)$. Theorem(4.16) shows also that $T_k \in B(X,Y)$ and $\|\{T_k\}\| \leq \|T\|$. Consequently, we can make the following definition.

Definition 4.4 *Suppose that X and Y are Banach spaces and Y contains no copy of c_0. For $T \in B(c(X), c(Y))$, define $\chi(T) \in B(X,Y)$ by*

$$\chi(T)(x) = \psi(\lim T)(x) = \lim T\mathbf{e}(x) - \sum_{k=1}^{\infty} \lim T\mathbf{e}_k(x), \quad \textit{for } x \in X.$$

For $n = 1, 2, \ldots$ define $\chi_n(T) \in B(X,Y)$ by

$$\chi_n(T)(x) = \psi(P_n T)(x) = P_n T\mathbf{e}(x) - \sum_{k=1}^{\infty} P_n T\mathbf{e}_k(x), \quad \textit{for } x \in X.$$

In other notation,

$$\chi(T)(x) = \lim T\mathbf{e}(x) - \sum_{k=1}^{\infty} T_k(x)$$

and

$$\chi_n(T)(x) = P_n T\mathbf{e}(x) - \sum_{k=1}^{\infty} T_{nk}(x).$$

Lemma 4.17 *Suppose X and Y are Banach spaces and Y contains no copy of c_0. If $T \in B(c(X), c(Y))$, then $\{\chi_n(T)\}$ forms a bounded sequence of operators in $B(X,Y)$.*

PROOF: Let $x \in X$ and $y_n = \sum_{k=1}^{\infty} T_{nk}(x)$. Then, since $T : c_0(X) \to l_\infty(Y)$ and the A_{nk} of Theorem(4.13) is the T_{nk} here, Corollary(4.15) shows that $\{y_n\}$ is bounded. Hence,

$$\|\chi_n(T)(x)\| \leq \|T\|\|x\| + \sup_{n\geq 1} \|y_n\| < \infty$$

for each $x \in X$. The Banach-Steinhaus theorem now provides a uniform bound for the $\chi_n(T)$. □

For a sequence of operators $\Lambda = \{\Lambda_k\}$ with $\Lambda_k \in B(X,Y)$ and a linear operator $L : s(X) \longrightarrow X$, define the operator $\Lambda \otimes L : s(X) \longrightarrow s(Y)$ by

$$\Lambda \otimes L(\mathbf{x}) = \{\Lambda_k L(\mathbf{x})\}$$

for $\mathbf{x} = \{x_k\} \in s(X)$. We shall use this notation in the case where $\Lambda_k = \chi_k(T)$ for $T \in B(c(X), c(Y))$ and $L = \lim \in B(c(X), X)$ where, in view of Lemma(4.17), if Y contains no copy of c_0, we obtain an operator from $c(X)$ to $l_\infty(Y)$.

We can now state the following theorem.

Theorem 4.18 *Let X and Y be Banach spaces with Y containing no copy of c_0. Given an operator $T \in B(c(X), c(Y))$, there is a unique sequence $\Lambda = \{\Lambda_k\} \in l_\infty(B(X,Y))$ and a unique matrix*

$$\mathbf{A} = \{A_{nk}\} \textit{ with } A_{nk} \in B(X,Y) \textit{ and } \mathbf{A} \in B(l_\infty(X), l_\infty(Y)),$$

such that for $\mathbf{x} = \{x_k\} \in c(X)$,

$$T\mathbf{x} = \Lambda \otimes \lim \mathbf{x} + \{\sum_{k=1}^{\infty} A_{nk} x_k\}. \tag{4.8}$$

Indeed,

$$\Lambda_k = \chi_k(T) \text{ and } A_{nk} = T_{nk} = P_n T \mathbf{e}_k. \tag{4.9}$$

Moreover,

$$\|T\| = \sup_{n \geq 1} \|\{\chi_n(T), T_{n1}, T_{n2}, \ldots\}\| \tag{4.10}$$

and

$$\lim T\mathbf{x} = \chi(T) \lim \mathbf{x} + \sum_{k=1}^{\infty} T_k x_k \tag{4.11}$$

where, for $k = 1, 2, \ldots$ and $x \in X$, $T_k x = \lim_{n\to\infty} T_{nk} x$.

PROOF: Let $\mathbf{x} = \{x_k\} \in c(X)$. Then

$$\mathbf{x} = \mathbf{e}(\lim \mathbf{x}) + \sum_{k=1}^{\infty} \mathbf{e}_k(x_k - \lim \mathbf{x}).$$

Thus,

$$P_n T\mathbf{x} = P_n T\mathbf{e}(\lim \mathbf{x}) - \sum_{k=1}^{\infty} P_n T\mathbf{e}_k(\lim \mathbf{x}) + \sum_{k=1}^{\infty} P_n T\mathbf{e}_k(x_k)$$

$$= \chi_n(T) \lim \mathbf{x} + \sum_{k=1}^{\infty} T_{nk}(x_k).$$

That $\{\chi_n(T)\} \in l_\infty(B(X,Y))$ is the content of Lemma(4.17). Clearly $T_{nk} \in B(X,Y)$. We have thus established (4.8) and (4.9). The uniqueness of the representation is shown by the following argument.

Suppose T is given by (4.8). then for $x \in X$,

$$P_n T\mathbf{e}_k(x) = \sum_{\nu=1}^{\infty} A_{n\nu} \mathbf{e}_k(x) = A_{nk}(x).$$

In other words, $A_{nk} = T_{nk}$. Also, for $x \in X$,

$$P_n T\mathbf{e}(x) = \Lambda_n(x) + \sum_{\nu=1}^{\infty} T_{n\nu} x$$

from which it follows that $\Lambda_n = \chi_n(T)$.

Now we establish (4.10). Let H denote the group norm in question. We first show that $H < \infty$. Suppose $x_0, x_1, \ldots, x_m \in B_X$. Then

$$\|\chi_n(T)(x_0) + \sum_{k=1}^{m} T_{nk}x_k\|$$

$$\leq \|P_nT\mathbf{x}\| + \|\sum_{k=m+1}^{\infty} T_{nk}x_0\|$$

$$\leq \|T\| + \|\{T_{nk}\}\| < \infty,$$

where $\mathbf{x} = \{x_1, x_2, \ldots, x_m, x_0, x_0, \ldots\}$ and the latter group norm is finite because $\mathbf{A} \in B(l_\infty(X), l_\infty(Y))$ by Corollary(4.15). Thus $H < \infty$. If $\mathbf{x} = \{x_k\} \in B_{c(X)}$, then $\|x_k\| \leq 1$ and $\|\lim \mathbf{x}\| \leq 1$, so

$$\|T\mathbf{x}\| = \sup_{n\geq 1} \|\chi_n(T)\lim(\mathbf{x}) + \sum_{k=1}^{\infty} T_{nk}x_k\| \leq H.$$

Hence, $\|T\| \leq H$. On the other hand, let $\epsilon > 0$. and choose n_0 so that

$$\|\{\chi_{n_0}(T), T_{n_01}, T_{n_02}, \ldots\}\| > H - \epsilon.$$

Now using Lemma(4.8), choose $f \in B_{Y^*}$ such that

$$\|\{f\chi_{n_0}(T), fT_{n_01}, fT_{n_02}, \ldots\}\| > H - \epsilon.$$

By Lemma(4.9), we have

$$\|f\chi_{n_0}(T)\| + \sum_{k=1}^{\infty} \|fT_{n_0k}\| > H - \epsilon.$$

Choose $x_0, x_1, \ldots, x_m \in B_X$ such that

$$f\chi_{n_0}(T)(x_0) + \sum_{k=1}^{m} fT_{n_0k}x_k > H - \epsilon$$

and then choose $l > m$ such that $\sum_{k=l+1}^{\infty} \|fT_{n_0k}\| < \epsilon$. Let

$$\mathbf{x} = \{x_1, x_2, \ldots, x_m, 0, 0, \ldots, \underbrace{0}_{l}, x_0, x_0, \ldots\}.$$

Then $\mathbf{x} \in B_{c(X)}$ and

$$\|T\mathbf{x}\| \geq \|P_{n_0}T\mathbf{x}\| \geq |fP_{n_0}T\mathbf{x}| > H - 2\epsilon.$$

Hence (4.10) holds.

Since $\mathbf{x} - \mathbf{e}(\lim \mathbf{x}) \in c_0(X)$, we have, from Theorem(4.16), that

$$\lim(T\mathbf{x} - T\mathbf{e}\lim \mathbf{x}) = \sum_{k=1}^{\infty} T_k(x_k - \lim \mathbf{x}).$$

Consequently, (4.11) holds. □

Remark 1. Since for $\mathbf{x} \in c_0(X)$ we have $T\mathbf{x} = \mathbf{A}x$ we see that $\mathbf{A} \in B(c_0(X), c(Y))$.

Remark 2. Given $\Lambda = \{\Lambda_k\} \in l_\infty(B(X,Y))$, let $\mathbf{A} = \operatorname{diag}\{-\Lambda_k\}$. Then if

$$T\mathbf{x} = \Lambda \otimes \lim \mathbf{x} + \mathbf{A}x$$

we see that $T \in B(c(X), c(Y))$.

Remark 3. In order that the matrix $\mathbf{A} = \{A_{nk}\} \in B(c(X), c(Y))$ (where Y contains no copy of c_0) it is necessary and sufficient that

(i) $\sup_{n\geq 1} \|\{A_{nk}\}\| < \infty$,

(ii) $\lim_{n\to\infty} A_{nk}(x)$ exists for $k = 1, 2, 3, \ldots$ and $x \in X$, and

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} A_{nk}(x)$ exists for $x \in X$ where the series converge unconditionally in norm in Y, see [36, page 38].

4.4 The Case $Y \supset c_0$.

The problem is more complicated when $Y \supset c_0$, because we no longer have unconditional convergence with which to work. Unconditional convergence has often to be *where $\chi(T) \in B(X, Y)$ is given by*

$$\chi(T)x = \phi(\lim \circ T) = \lim T\mathbf{e}(x) - \sum_{k=1}^{\infty}{}^{\sigma} \lim T\mathbf{e}_k x.$$

It is somewhat annoying to find condition (4.12) appearing in addition to the required representability of the operator T. It is, however, necessary as the following example shows.

Example 4.1

Let $X = \mathbb{C}$ and $Y = c_0$. Define $T : s(\mathbb{C}) \to s(c_0)$ by

$$T = \begin{pmatrix} e_1 & -e_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ e_1 & e_2 & -e_1 & -e_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ e_1 & e_2 & e_3 & -e_1 & -e_2 & -e_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ e_1 & e_2 & e_3 & e_4 & -e_1 & -e_2 & -e_3 & -e_4 & 0 & 0 & 0 & \cdots \\ e_1 & e_2 & e_3 & e_4 & e_5 & -e_1 & -e_2 & -e_3 & -e_4 & -e_5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then $T \in B(c, c(c_0))$, T is representable because it is row finite, but $\sum_k T_k 1 = \sum_k e_k$ is not weakly convergent in c_0. To see that $T \in B(c, c(c_0))$, consider $\mathbf{x} = \{x_k\} \in c$. We have

$$P_n T\mathbf{x} = \sum_{i=1}^{n} (x_i - x_{n+i}) e_i \in c_0.$$

Let $\mathbf{y} = \{y_j\} = \mathbf{x} - (\lim \mathbf{x})e \in c_0$. Then $T\mathbf{x} \in c(c_0)$ with $\mathbf{y}$ as limit. Indeed

$$\|P_n T\mathbf{x} - \mathbf{y}\| = \sup_j |(P_n T\mathbf{x})_j - y_j|$$

$$\leqslant \max\left(\sup_{j \leqslant n} |x_j - x_{n+j} - x_j + \lim \mathbf{x}|, \quad \sup_{j>n} |x_j - \lim \mathbf{x}|\right)$$

$$= \max\left(\sup_{j \leqslant n} |x_{n+j} - \lim \mathbf{x}|, \quad \sup_{j>n} |x_j - \lim \mathbf{x}|\right) = \sup_{j>n} |x_j - \lim \mathbf{x}| \to 0,$$

as $n \to \infty$. Therefore $T\mathbf{x} = \{P_n T\mathbf{x}\}_n \in c(c_0)$, and it is clear that $\|T\| \leqslant 2$.

replaced by weak convergence. Even then Cass showed that when $Y \supset c_0$ there are always operators T which cannot be represented in a way analogous to Theorem(4.18). Cass gave the following definition.

Definition 4.5 *Suppose X and Y are banach spaces. An operator $T \in B(c(X), c(Y))$ is* **representable** *if $\sum_{k=1}^{\infty} T_{nk}x$ is weakly convergent for all $x \in X$, where $T_{nk} = P_n T \mathbf{e}_k \in B(X, Y)$.*

Cass then proved the following theorem and Corollary(4.20).

Theorem 4.19 *Let X and Y be Banach spaces. Given a representable operator $T \in B(c(X), c(Y))$, there is a unique sequence $\Lambda = \{\Lambda_k\} \in l_\infty(B(X, Y))$ and a unique matrix $\mathbf{A} = \{A_{nk}\}$ with $A_{nk} \in B(X, Y)$ and $\mathbf{A} \in B(c_0(X), c(Y))$ such that for $\mathbf{x} = \{x_k\} \in c(X)$,*

$$Tx = \Lambda \otimes \lim \mathbf{x} + \{\sum_{k=1}^{\infty}{}^{\sigma} A_{nk} x_k\}.$$

Indeed

$$\Lambda_k = \chi_k(T) \text{ and } A_{nk} = T_{nk} = P_n T \mathbf{e}_k.$$

Moreover

$$\|T\| = \sup_{n \geqslant 1} \|\{\chi_n(T), T_{n1}, T_{n2}, \ldots\}.\|$$

Corollary 4.20 *Let $T \in B(c(X), c(Y))$ be representable and suppose*

$$\sum_{k=1}^{\infty} \lim_{n \to \infty} T_{nk}(x) = \sum_{k=1}^{\infty} T_k x \text{ converges weakly for } x \in X. \tag{4.12}$$

Then

$$\lim T\mathbf{x} = \chi(T) \lim \mathbf{x} + \sum_{k=1}^{\infty}{}^{\sigma} T_k x_k,$$

where $\chi(T) \in B(X,Y)$ *is given by*

$$\chi(T)x = \phi(\lim \circ T) = \lim T\mathbf{e}(x) - \sum_{k=1}^{\infty}{}^{\sigma} \lim T\mathbf{e}_k x.$$

It is somewhat annoying to find condition (4.12) appearing in addition to the required representability of the operator T. It is, however, necessary as the following example shows.

Example 4.1

Let $X = \mathbb{C}$ and $Y = c_0$. Define $T : s(\mathbb{C}) \to s(c_0)$ by

$$T = \begin{pmatrix} e_1 & -e_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ e_1 & e_2 & -e_1 & -e_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ e_1 & e_2 & e_3 & -e_1 & -e_2 & -e_3 & 0 & 0 & 0 & 0 & 0 & \cdots \\ e_1 & e_2 & e_3 & e_4 & -e_1 & -e_2 & -e_3 & -e_4 & 0 & 0 & 0 & \cdots \\ e_1 & e_2 & e_3 & e_4 & e_5 & -e_1 & -e_2 & -e_3 & -e_4 & -e_5 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Then $T \in B(c, c(c_0))$, T is representable because it is row finite, but $\sum_k T_k 1 = \sum_k e_k$ is not weakly convergent in c_0. To see that $T \in B(c, c(c_0))$, consider $\mathbf{x} = \{x_k\} \in c$. We have

$$P_n T\mathbf{x} = \sum_{i=1}^{n} (x_i - x_{n+i}) e_i \in c_0.$$

Let $\mathbf{y} = \{y_j\} = \mathbf{x} - (\lim \mathbf{x})e \in c_0$. Then $T\mathbf{x} \in c(c_0)$ with $\mathbf{y}$ as limit. Indeed

$$\|P_n T\mathbf{x} - \mathbf{y}\| = \sup_j |(P_n T\mathbf{x})_j - y_j|$$

$$\leqslant \max\left(\sup_{j \leqslant n} |x_j - x_{n+j} - x_j + \lim \mathbf{x}|, \quad \sup_{j > n} |x_j - \lim \mathbf{x}|\right)$$

$$= \max\left(\sup_{j \leqslant n} |x_{n+j} - \lim \mathbf{x}|, \quad \sup_{j > n} |x_j - \lim \mathbf{x}|\right) = \sup_{j > n} |x_j - \lim \mathbf{x}| \to 0,$$

as $n \to \infty$. Therefore $T\mathbf{x} = \{P_n T\mathbf{x}\}_n \in c(c_0)$, and it is clear that $\|T\| \leqslant 2$.

Chapter 5

Generalized Köthe-Toeplitz Duals on Banach Sequence Spaces

In this chapter generalized Köthe-Toeplitz duals on Banach Spaces are defined and some previous results are summarized. Then we will refine some of the inclusion results in [18] and show that if Y contains an isomorphic copy of c_0, then for any Banach space X, all generalized Köthe-Toeplitz duals of $bv(X)_Y$ are distinct.

5.1 Introduction

Definition 5.1 Generalized Köthe-Toeplitz Duals

Let X and Y be Banach spaces and E be a nonempty subset of $s(X)$. In the following, $\{T_k\}$ will always denote a sequence of operators in $B(X,Y)$.

$$E^{\alpha} = \left\{ \{T_k\} \;\middle|\; \sum_{k=1}^{\infty} \|T_k x_k\| < \infty \;\; \textit{for all } \{x_k\} \in E \right\},$$

$$E^{v} = \left\{ \{T_k\} \;\middle|\; \sum_{k=1}^{\infty} T_k x_k \;\; \textit{converges unconditionally for all } \{x_k\} \in E \right\},$$

$$E^{\beta} = \left\{ \{T_k\} \;\middle|\; \sum_{k=1}^{\infty} T_k x_k \;\; \textit{converges for all } \{x_k\} \in E \right\},$$

$$E^{\sigma} = \left\{ \{T_k\} \;\middle|\; \sum_{k=1}^{\infty} T_k x_k \;\; \textit{converges weakly for all } \{x_k\} \in E \right\},$$

$$E^{\sigma_c} = \left\{ \{T_k\} \;\middle|\; \sum_{k=1}^{\infty} T_k x_k \text{ is weakly Cauchy for all } \{x_k\} \in E \right\},$$

$$E^{\gamma} = \left\{ \{T_k\} \;\middle|\; \sup_{n \geq 1} \|\sum_{k=1}^{n} T_k x_k\| < \infty \text{ for all } \{x_k\} \in E \right\},$$

$$E^{WUC_p} = \left\{ \{T_k\} \;\middle|\; \sum_{k=1}^{\infty} |y^* T_k x_k|^p < \infty \text{ for all } y^* \in Y^* \text{ and all } \{x_k\} \in E \right\},$$

where $1 \leq p < \infty$,

$$E_{\infty} = \left\{ \{T_k\} \;\middle|\; \sup_{k \geq 1} \|T_k x_k\| < \infty \text{ for all } \{x_k\} \in E \right\}.$$

These duals clearly depend on X and Y. It seems needless, however, to incorporate this dependence into the notation except when confusion would otherwise arise or emphasis is needed, in which case we use notation like $E(X)_Y$.

Among the duals above, Maddox defined E^{α} and E^{β} in [36], Thorp defined E^{β}, E^{σ} and E^{σ_c} in [45] using different notation. The rest of the duals were defined by Cass in [18]. Although Maddox allows for the possibility of unbounded operators in his definitions, we follow Cass in requiring all operators T_k to be bounded, for reasons similar to those stated in §4.1, when we defined the matrix class (E, F).

The following inclusions are easy to prove.

$$E^{\alpha} \subset E^{v} \subset E^{\beta} \subset E^{\sigma} \subset E^{\sigma_c} \subset E^{\gamma} \subset E_{\infty}, \tag{5.1}$$

$$E^{v} \subset E^{WUC_p} \subset E^{WUC_q} \subset E_{\infty} \text{ for } 1 \leq p < q, \tag{5.2}$$

$$E^{WUC_1} \subset E^{\sigma_c}. \tag{5.3}$$

Maddox [36, page 21f] shows that both of the inclusions $c_0^{\beta}(X) \supset c^{\beta}(X)$ and $c^{\beta}(X) \supset l_{\infty}^{\beta}(X)$ may be strict. For his examples, he takes $Y = l_{\infty}$ and c_0 respectively. This result in [19], however, shows that the above strict inclusions depend on the condition that Y contains a copy of c_0.

Theorem 5.1 *Let X and Y be Banach spaces and suppose Y contains no copy of c_0. Then*

$$c_0^\upsilon(X) = c_0^\beta(X) = c_0^\sigma(X) = c^\upsilon(X) = c^\beta(X) = c^\sigma(X) = l_\infty^\upsilon(X) = l_\infty^\beta(X) = l_\infty^\sigma(X).$$

PROOF: That $c_0^\beta(X) \supset c^\beta(X) \supset l_\infty^\beta(X)$ and $c_0^\upsilon(X) \supset c^\upsilon(X) \supset l_\infty^\upsilon(X)$ is clear. Lemma(4.7) shows that $c_0^\beta(X) \subset l_\infty^\upsilon(X)$. Hence, $c_0^\beta(X) = c^\beta(X) = l_\infty^\beta(X)$ and $l_\infty^\upsilon(X) = l_\infty^\beta(X)$. Also, $c_0^\upsilon(X) = c_0^\beta(X)$ and now $l_\infty^\upsilon(X) = c_0^\upsilon(X)$. Lemma(4.5) shows that $c_0^\sigma(X) = c_0^\upsilon(X)$ and $l_\infty^\sigma(X) = l_\infty^\upsilon(X)$. Since $c_0^\sigma(X) \supset c^\sigma(X) \supset l_\infty^\sigma(X)$, we also have $c^\sigma(X) = c_0^\sigma(X) = c^\upsilon(X)$, and the result follows. □

Equality with the α-dual, under the hypotheses of Theorem(5.1), may fail. For, choose $y_n \in l_1$ so that $\sum y_n$ converges unconditionally in norm in l_1 and $\|y_n\| = 1/n$ [33, page 16]; take $T_k x = x y_k$ for $x \in \mathbf{R}$. Then $\{T_k\} \in c_0^\upsilon \setminus c_0^\alpha$.

Further studies of inclusion relationships between different duals were done by Cass in [18]. We will not list results proved in [18] here except Theorem(5.3), which I helped prove.

Definition 5.2 *For Banach spaces X and Y and $1 \leqslant p < \infty$, an operator $T \in B(X,Y)$ is said to be* **absolutely p-summing** *if $\sum_{k=1}^\infty \|Tx_k\|^p < \infty$ whenever*

$$\sum_{k=1}^\infty |x^*(x_k)|^p < \infty \text{ for all } x^* \in X^*.$$

Theorem(5.2) is a variation of the Dvoretsky-Rogers Theorem. Its proof can be found, for example, on page 61 of [24].

Theorem 5.2 *If $1 \leqslant p < \infty$, and Y is infinite-dimensional, then the identity operator on Y is not absolutely p-summing.*

Theorem 5.3 *Let X and Y be Banach spaces with $\dim Y = \infty$ and $1 < p < \infty$, then $l_p^\alpha(X) \neq l_p^\upsilon(X)$.*

Proof: Since $\dim Y = \infty$, the identity operator is not absolutely p'-summing by Theorem(5.2). Thus there is a sequence $\{y_k\} \in Y$ such that $\sum_{k=1}^{\infty} \|y_k\|^{p'} = \infty$ while $\sup_{y \in B_{Y^*}} \sum_{k=1}^{\infty} |y^*(y_k)|^{p'} < \infty$. Define $A_k \in B(\mathbf{C}, Y)$ by $A_k(1) = y_k$. Take $x \in B_X$, $x \neq 0$ and $f \in B_{X^*}$ such that $f(x) = 1$. Let $T_k = A_k \circ f$. Using Propositions 3.8 and 3.9 in [36] we find that $\{T_k\} \in l_p^{\beta}(X) = l_p^{\nu}(X)$, but $\{T_k\} \notin l_p^{\alpha}(X)$. □

5.2 Strict Inclusion of Generalized Köthe-Toeplitz Duals

In this section we focus on the case of Y containing an isomorphic copy of c_0. We first clarify the relationships between duals of $c(X)$, $c_0(X)$ and $l_\infty(X)$.

Theorem 5.4 *Let X and Y be Banach spaces and suppose that Y contains an isomorphic copy of c_0. Then the following table of equalities and inclusions holds.*

$$
\begin{array}{ccccc}
l_\infty^{\alpha}(X) & = & c^{\alpha}(X) & = & c_0^{\alpha}(X) \\
\cap\nparallel & & \cap\nparallel & & \cap\nparallel \\
l_\infty^{\nu}(X) & ? & c^{\nu}(X) & \subsetneqq & c_0^{\nu}(X) \\
\| & & \cap\nparallel & & \| \\
l_\infty^{\beta}(X) & \subsetneqq & c^{\beta}(X) & \subsetneqq & c_0^{\beta}(X) \\
\| & I & \cap\nparallel & & \| \\
l_\infty^{\sigma}(X) & \subsetneqq & c^{\sigma}(X) & \subsetneqq & c_0^{\sigma}(X) \\
\cap\nparallel & & \cap\nparallel & II & \| \\
l_\infty^{\gamma}(X) & = & c^{\gamma}(X) & = & c_0^{\gamma}(X)
\end{array}
$$

There are examples where $l_\infty^{\nu}(X) = c^{\nu}(X)$ and examples where $l_\infty^{\nu}(X) \subsetneqq c^{\nu}(X)$.

Proof: Rows 1, 3, and 5 of the table follow from [36, Proposition 3.4, p. 23], [18, Theorem 13], and [18, Theorem 7] respectively. It is shown in [45] that $l_\infty^{\alpha}(X) \subsetneqq$

$l_\infty^\beta(X) = l_\infty^v(X) = l_\infty^\sigma(X)$ Hence, we also have $l_\infty^\sigma(X) \subsetneqq l_\infty^\gamma(X)$ from [18, Theorem 7, (11), and Corollary 2]. Thus the left hand column of the table holds. The central column follows from [18, Theorems 11, 9, and 13]. The right hand column is a consequence of [36, Proposition 3.4], [45] and [18, Theorem 7].

Suppose $c^v(X) = c_0^v(X)$. Then $c^v(X) = c_0^\beta(X) \supsetneqq c^\beta(X) \supsetneqq c^v(X)$ which is impossible. One shows similarly, using squares I and II that $l_\infty^\sigma(X) \subsetneqq c^\sigma(X) \subsetneqq c_0^\sigma(X)$.

To see that it is possible to have $l_\infty^v(X) = c^v(X)$, let $X = \mathbb{C}$ and Y be arbitrary. Let $\{T_k\} \in c^v(X)$ and $y_k = T_k(1)$. Then $\sum_{k=1}^\infty y_k = \sum_{k=1}^\infty T_k(1)$ is unconditionally convergent. Now let $\mathbf{z} = \{z_k\} \in l_\infty(X) = l_\infty$ We have $\sum_{k=1}^\infty T_k z_k = \sum_{k=1}^\infty z_k y_k$ which is unconditionally convergent ([42, p. 458 1° $\Leftrightarrow$ 5°]). Thus $\{T_k\} \in l_\infty^v(X)$. Since $l_\infty^v(X) \subset c^v(X)$ always, our result follows.

To see that it is possible to have $l_\infty^v(X) \subsetneqq c^v(X)$, let $X = Y = c_0$. For $i \geq 1$, define $T_i x = \xi_i e_i$ where $x = \{\xi_i\} \in c_0$. For any sequence $\mathbf{x} = \{x_k\} = \{\{\xi_{kj}\}\} \in c(c_0)$, we have $\xi_{kk} \to 0$ as $k \to \infty$. Indeed, suppose $x_k \to x = \{\xi_j\} \in c_0$. Then, given $\epsilon > 0, \exists K$ such that for $k \geq K$, $\|x_k - x\|_\infty = \sup_{j\geq 1} |\xi_{kj} - \xi_j| < \epsilon/2$. Also, $\exists J$ such that for $j \geq J$, $|\xi_j| < \epsilon/2$. Hence, if $k \geq \max(K, J)$, then $|\xi_{kk}| \leq |\xi_{kk} - \xi_k| + |\xi_k| < \epsilon$. Consequently, the series

$$\sum_{k=1}^\infty T_k x_k = \sum_{k=1}^\infty \xi_{kk} e_k$$

converges unconditionally to $\{\xi_{kk}\} \in c_0$. Thus $\{T_k\} \in c^v(X)$. On the other hand, $\{T_k\} \notin l_\infty^v(X)$, because, for example, $\{e_k\} \in l_\infty(c_0)$ but $\sum_{k=1}^n T_k e_k = \sum_{k=1}^n e_k$, which is not convergent in c_0. □

We now examine the generalized Köthe-Toeplitz duals of $bv(X)$ and show that when Y is a Banach space containing an isomorphic copy of c_0, all of the duals are different. We also look at the special cases $Y = l_1$ and $Y = l_2$, to give an idea of what can happen to the duals when Y does not contain an isomorphic copy of c_0. We recall that

$$bv(X) = \left\{ \mathbf{x} = \{x_\kappa\} \in s(X) : \|\mathbf{x}\|_{bv} = \|x_1\| + \sum_{k=1}^{\infty} \|x_k - x_{k+1}\| < \infty \right\}.$$

For $\{T_k\} \subset B(X,Y)$, we have the following version of Abel's partial summation formula. Let $\mathcal{T}_m = \sum_{k=1}^{m} T_k$. Then for $x_1, x_2, \ldots, x_n \in X$, we have

$$\sum_{k=1}^{n} T_k x_k = \sum_{k=1}^{n-1} \mathcal{T}_k(x_k - x_{k+1}) + \mathcal{T}_n x_n, \text{ for } n = 1,2,\ldots \tag{5.4}$$

Proposition 5.1 *Let X and Y be Banach spaces. Then*

$$bv^{\nu}(X) = \left\{ \{T_k\} : \sum_{k=1}^{\infty} T_k \textit{ converges unconditionally in the strong operator topology} \right\} \tag{5.5}$$

$$bv^{\beta}(X) = \left\{ \{T_k\} : \sum_{k=1}^{\infty} T_k \textit{ converges in the strong operator topology} \right\} \tag{5.6}$$

$$bv^{\sigma}(X) = \left\{ \{T_k\} : \sum_{k=1}^{\infty} T_k \textit{ converges in the weak operator topology} \right\} \tag{5.7}$$

$$bv^{WUC_1}(X) = \left\{ \{T_k\} : \sum_{k=1}^{\infty} T_k \textit{ is weakly unconditionally Cauchy for all } x \in X \right\} \tag{5.8}$$

$$bv^{\sigma_c}(X) = \left\{ \{T_k\} : \sum_{k=1}^{\infty} T_k x \textit{ is weakly Cauchy for all } x \in X \right\} \tag{5.9}$$

$$bv^{\gamma}(X) = \left\{ \{T_k\} : \sup_{n \geq 1} \| \sum_{k=1}^{n} T_k \| < \infty \right\} = bs(B(X,Y)) \tag{5.10}$$

$$bv_{\infty}(X) = \left\{ \{T_k\} : \sup_{n \geq 1} \|T_n\| < \infty \right\} = l_{\infty}(B(X,Y)) \tag{5.11}$$

PROOF: Because the constant sequence, $\{x\} \in bv(X)$ for all $x \in X$, we see that the left side of each of (5.5)–(5.9) is contained in the corresponding right side.

Suppose $\sum_{k=1}^{\infty} T_k$ converges unconditionally in the strong operator topology. Let $\mathbf{x} = \{x_k\} \in bv(X)$. For an increasing sequence $\{k_n\}$ of positive integers and $m \geq 1$, we have, using (5.4),

$$\sum_{n=1}^{m} T_{k_n} x_{k_n} = \sum_{n=1}^{m-1} S_n(x_{k_n} - x_{k_{n+1}}) + S_m x_{k_m},$$

where $S_n = \sum_{i=1}^{n} T_{k_i}$.

Because $\sum_{k=1}^{\infty} T_k$ converges unconditionally in the strong topology, the Banach-Steinhaus theorem shows that $\sup_{n\geq 1} \|S_n\| < \infty$. Also $\sum_{n=1}^{\infty} \|x_{k_n} - x_{k_{n+1}}\| \leq \sum_{k=1}^{\infty} \|x_k - x_{k+1}\| < \infty$. Thus $\sum_{n=1}^{\infty} S_n(x_{k_n} - x_{k_{n+1}})$ is absolutely convergent in Y. Moreover, letting x be the norm limit of $\{x_k\}$ and S be the limit of $\{S_k\}$ in the strong operator topology, we have

$$S_m x_{k_m} = S_m x + S_m(x_{k_m} - x) \to Sx \text{ as } m \to \infty.$$

Thus $\sum_{n=1}^{\infty} T_{k_n} x_{k_n}$ converges in Y, and so $\{T_k\} \in bv^{u}(X)$.

That the right side of (5.6) is contained in the left side is proved similarly and, in fact, more simply, because one has only to consider the case of the sequence $\{k_n\}$ with $k_n = n$.

To complete the proof of (5.7), we suppose $\sum_{k=1}^{\infty} T_k$ converges to $T \in B(X, Y)$ in the weak operator topology and take any sequence $\{x_k\} \in bv(X)$ and any natural number n. The Banach-Steinhaus Theorem shows that $\{T_k\}$ is bounded in $B(X, Y)$. So as before, $\sum_{k=1}^{\infty} T_k(x_k - x_{k+1})$ is absolutely convergent. Also $T_n x_n \to Tx$ weakly where x is the norm limit of $\{x_k\}$. This proves that $\{T_k\} \in bv^{\sigma}(X)$ so that (5.7) holds.

To finish the proof of (5.8) we proceed as in the proof of (5.5). Suppose $\sum_{k=1}^{\infty} T_k x$ is weakly unconditionally Cauchy for all $x \in X$. Take $\{x_n\} \in bv(X)$, an increasing sequence of positive integers $\{k_n\}$, and any $f \in Y^*$. For any natural number m, we have,

$$\sum_{n=1}^{m} f(T_{k_n} x_{k_n}) = \sum_{n=1}^{m-1} g_n(x_{k_n} - x_{k_{n+1}}) + g_m x_{k_m},$$

where $g_n = \sum_{i=1}^{n} f \circ T_{k_n}$. Now, for similar reasons to those adduced in the proof of (5.5), $\sum_{n=1}^{\infty} f(T_{k_n} x_{k_n})$ is convergent. Hence $\{T_k\} \in bv^{WUC_1}(X)$. The proof of (5.9) is similar to that of (5.7).

To prove (5.10) suppose that $\{T_k\} \in bv^{\gamma}(X)$ and that $\{T_k\}$ is unbounded in $B(X, Y)$. By the Banach-Steinhaus Theorem, there is an $x_0 \in X$ such that $\{T_k x_0\}$ is unbounded. Taking the constant sequence $\{x_0\} \in bv(X)$ we see that $\{T_k\} \notin bv^{\gamma}(X)$. Hence, $bv^{\gamma}(X) \subset bs(B(X, Y))$. On the other hand, if $\{T_k\} \in bs(B(X, Y))$, then for $\{x_k\} \in bv(X)$, we see from (5.4), that $\{\sum_{k=1}^{n} T_k x_k\}$ is bounded. Thus $bs(B(X, Y)) \subset bv^{\gamma}(X)$, which completes the proof of (5.10).

Finally, it is clear that $l_{\infty}(B(X, Y)) \subset bv_{\infty}(X)$, and the reverse inclusion follows from the Banach-Steinhaus Theorem. □

Let E^{ξ} denote an unspecified Köthe-Toeplitz dual of E. For $\{T_k\} \in E^{\xi}$ we say that $\{y_k\} = \{T_k x_k\}$ satisfies the **ξ-condition** for $\{x_k\} \in E$. We extend this notation in the obvious way to say that a sequence $\{y_k\} \subset Y$ satisfies the ξ-condition even when there is no explicit mention of any sequence of operators $\{T_k\}$.

Lemma 5.5 *Suppose Y and Z are Banach spaces and B is an isomorphism from Y onto a (closed) subspace of Z. Then a sequence $\{y_k\} \subset Y$ satisfies the ξ-condition if and only if $\{By_k\} \subset Z$ satisfies the ξ-condition.*

PROOF: The various cases that arise in this proof are easily managed with the aid of the Hahn-Banach Theorem and the facts (i), that B is continuous and (ii), that there is a number $\delta > 0$ such that $\|By\| \geq \delta \|y\|$ for all $y \in Y$. □

Lemma 5.6 *Suppose X, X_1, Y, and Y_1 are Banach spaces, and B is an isomorphic mapping from Y_1 to a (closed) subspace of Y. Let $A \in B(X, X_1)$ and $E \subset s(X)$. For $\mathbf{x} = \{x_k\} \in s(X)$, we write $F_A(\mathbf{x}) = \{Ax_k\}$. If $(F_A E)^{\xi}_{Y_1} \neq (F_A E)^{\zeta}_{Y_1}$, then $E^{\xi}_{Y} \neq E^{\zeta}_{Y}$.*

Here ξ and ζ signify any two different Köthe-Toeplitz duals of the set in question.

PROOF: Suppose without loss of generality that $\{T_k\} \in (F_A E)^{\xi}_{Y_1} \setminus (F_A E)^{\zeta}_{Y_1}$. Define $S_k = B \circ T_k \circ A$. We show that $S_k \notin E^{\xi}_Y \setminus E^{\zeta}_Y$.

Now, for $\{x_k\} \in E$, $\{T_k A x_k\}$ satisfies the ξ-condition in Y_1. Applying Lemma(5.5) with Z replaced by Y and Y replaced by Y_1, we see that $\{BT_k A x_k\} = \{S_k x_k\}$ also satisfies the ξ-condition in Y. Hence $\{S_k\} \in E^{\xi}_Y$.

Because $\{T_k\} \notin (F_A E)^{\zeta}_{Y_1}$, there is a sequence $\{x_k\} \in E$ such that $\{T_k A x_k\}$ does not satisfy the ζ-condition in Y_1. Again applying Lemma(5.5), we find that $\{BT_k A x_k\}$ does not satisfy the ζ-condition in Y. $\square$

We now show the existence of sequence spaces where all the duals are different.

Theorem 5.7 *Let X be any Banach space and let Y be a Banach space which contains an isomorphic copy of c_0. Then all the generalized Köthe-Toeplitz duals of $bv(X)_Y$ are distinct.*

PROOF: Take a non-zero functional $f \in X^*$. It is readily seen that $F_f bv(X) = bv(\mathbf{C})$. From Lemma(5.6) it is enough to prove the theorem for $X = \mathbf{C}$ and $Y = c_0$.

In the case where $X = Y = \mathbf{C}$, it is known that $bv^{\alpha} = l_1$, $bv^{\beta} = cs$, $bv^{\gamma} = bs$, $bv^{WUC_p} = l_p$, and $bv_{\infty} = l_{\infty}$. If we then apply Lemma(5.6) with $X = X_1 = Y_1 = \mathbf{C}$, $Y = c_0$ $Az = z$, and $Bz = ze_1$, take account of [18, Theorem 1], and note (5.1), (5.2), and (5.3), we reduce the proof to a consideration of the following cases when $Y = c_0$:

$$bv^{\alpha}(\mathbf{C}) \subsetneqq bv^{\nu}(\mathbf{C}) \subsetneqq bv^{WUC_1}(\mathbf{C})$$

and

$$bv^{\beta}(\mathbf{C}) \subsetneqq bv^{\sigma}(\mathbf{C}) \subsetneqq bv^{\sigma_c}(\mathbf{C}).$$

If we take $T_k : \mathbb{C} \to c_0$ by $T_k(z) = (z/k)e_k$, we see that $bv^{\alpha}(\mathbb{C}) \subsetneqq bv^{\nu}(\mathbb{C})$. Take $T_k z = ze_k$ to show that $bv^{\nu}(\mathbb{C}) \subsetneqq bv^{WUC_1}(\mathbb{C})$. Take $T_k z = ze_k - ze_{k+1}$ to show that $bv^{\beta}(\mathbb{C}) \subsetneqq bv^{\sigma}(\mathbb{C})$. Finally, $T_k(z) = ze_k$ provides an example to show that $bv^{\sigma}(\mathbb{C}) \subsetneqq bv^{\sigma_c}(\mathbb{C})$. $\square$

When Y does not contain an isomorphic copy of c_0, it follows from a theorem of Bessaga and Pełczyński, [3, Theorem 5] (see also [18, Theorem 5]) that for any $E \subset s(X)$ we have $E^{WUC_1} = E^{\nu}$. This, however, may not be the only equality among the duals of $bv(X)$. If $Y = l_1$, then the coincidence of weak and strong sequential convergence and the fact that l_1 is weakly sequentially complete, yields $bv^{\beta}(X) = bv^{\sigma}(X) = bv^{\sigma_c}(X)$. On the other hand, if we take $Y = l_2$, which is reflexive, we find that $bv^{WUC_1}(X) = bv^{\nu}(X)$ and $bv^{\sigma}(X) = bv^{\sigma_c}(X)$ are the only equalities among the duals.

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